Fourier Facts

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Here is a list of facts (without proof) about Fourier analysis.

1. Suppose f is integrable (Riemann or Lebesgue) on $[-\pi, \pi]$ and $\hat{f}(n) = 0$ for all n. Then $f = 0$ almost everywhere (almost everywhere means except on a set of measure 0). If f is continuous then f is identically 0. Biefly
$$\widehat{f}(n) = 0 \text{ for all } n \implies f = 0 \text{ a.e.}$$

2. Let $S_N(x) = \sum_{-N}^{N} \widehat{f}(n) e^{inx}$, where $f \in L^2([-\pi, \pi])$. Then $\|f - S_N\|_2 \to 0$ as $n \to \infty$.

3. If $f, g \in L^2([-\pi, \pi])$ then $< f, g >= 2\pi \sum_{-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}}(n)$. Hence
$$\|f\|^2 = 2\pi \sum_{-\infty}^{\infty} |\widehat{f}(n)|^2$$

4. If $\sum_{-\infty}^{\infty} |c_n|^2 < \infty$ then there is $f \in L^2([-\pi, \pi])$ so that $\widehat{f}(n) = c_n$ (and $\|f - S_N\|_2 \to 0$ as $n \to \infty$). (Riesz-Fischer theorem)

5. Let R^1 be the set of Riemann integrable functions on $[-\pi, \pi]$ amd $R^2 = \{f : |f|^2 \in R^1\}$. We have proved $R^1 \subset R^2$. Let L^1, L^2 be defined similarly for Lebesgue integration. It's a theorem that $L^2 \subset L^1$. If $f \in R^1$ or $f \in L^1$ then $\hat{f}(n)$ is defined and the following is true (Riemann-Lebesgue lemma)
$$f \in R^1 \text{ or } f \in L^1 \implies \hat{f}(n) \to 0 \text{ as } n \to \infty$$

6. Let $C^{k+} = \{f \in C^k : f^{(k)} \text{ is piecewise smooth}\}$. (This means that $f^{(k+1)}$ exists and is continuous except at finitely many points and at those points $f^{(k+1)}$ has left and right limits.) Note that $C^{k+1} \subset C^{k+}$.
$$f \in C^{k+} \implies \widehat{f^{(k+1)}}(n) = (in)^{k+1}\widehat{f}(n)$$
Nothing is said here about convergence. Also
$$f \in C^{k+1} \implies \widehat{f^{(k+1)}}(n) = (in)^{k+1}\widehat{f}(n)$$

7. If $f \in L^1$ or $f \in R^1$ and $n^{k+1+\epsilon}\widehat{f}(n) \to 0$, where $\epsilon > 0$, then $f \in C^k$. Briefly,
$$n^{k+1+\epsilon}\widehat{f}(n) \to 0 \implies f \in C^k$$