Jensen's Inequality

One of the main tools for proving inequalities is Jensen's inequality. It often refers to an integral inequality, one form of which is

$$f\left(\int_0^1 g(x)\,dx\right) \le \int_0^1 f(g(x))\,dx$$

where f is a convex function. The "disrete" version is simpler, and almost trivial.

Def: Let f be real-valued function defined on an interval $I \subset \mathbb{R}$. f is convex if for every $p, q \ge 0$, with $p+q=1$, $f(px+qy) \le pf(x)+qf(y)$ for any $x, y \in I$.

Jensen's Inequality. Let f be convex on I and suppose $x_1, \ldots, x_n \in I$, $p_1, p_2, \ldots, p_n \ge 0$, $\sum_1^n p_j = 1$. Then

$$\text{(J)} \qquad f\left(\sum_1^n p_j x_j\right) \le \sum p_j f(x_j)$$

Proof: By induction on n. It's true for $n=2$. Suppose $p_j > 0$ for all j. Let $p = \sum_1^{n-1} p_j$, $s_j = \frac{p_j}{p}$ for $1 \le j \le n-1$, $q = p_n$, $x = \sum s_j x_j$, $y = x_n$, $\sum s_j = 1$.

Then $p+q=1$, $p>0$, $q>0$, $x, y \in I$, $px+qy = \sum_1^n p_j x_j$.

$$f\left(\sum p_j x_j\right) = f(px+qy) \le p f(x) + q f(y) = p f(x) + p_n f(x_n)$$
$$\le p \sum s_j f(x_j) + p_n f(x_n)$$
$$= \sum_1^n p_j f(x_j) .$$

QED

How to apply it?

<u>Theorem</u>. Let $f \in C^2(I)$, $f''(x) \ge 0$. Then f is convex.

<u>Proof</u>: In the previous notation, let $a = px + qy$

By Taylor's theorem:

$$f(x) = f(a) + f'(a)(x-a) + f''(\xi)(x-a)^2$$
$$f(y) = f(a) + f'(a)(y-a) + f''(\eta)(y-a)^2$$

where ξ is between x and a, η is between y and a.

Multiply the first equation by p, the second by q and add.

$$p f(x) + q f(y) = f(a) + p f''(\xi)(x-a)^2 + q f''(\eta)(y-a)^2$$
$$\ge f(px+qy),$$

using $p+q=1$, $p, q \ge 0$, and $f'' \ge 0$.

<u>APPLICATION</u>

$f(x) = -\log(x)$, on $x > 0$.

$f'(x) = -1/x$, $f''(x) = 1/x^2 > 0$.

So f is convex.

Hence if $x_j > 0$, $j = 1, \ldots, n$, $p_j \geq 0$, $\sum p_n = 1$,

$$-\log\left(\sum_1^n p_j x_j\right) \leq -\sum_1^n p_j \log(x_j) = -\log\left(x_1^{p_1} \cdots x_n^{p_n}\right)$$

Change signs and exponentiate:

$$x_1^{p_1} \cdots x_n^{p_n} \leq \sum_1^n p_j x_j ,$$

the <u>geometric mean</u> - <u>arithmetic mean</u> <u>inequality</u>.