

Collisions and Spirals of Loewner Traces

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Abstract

We analyze Loewner traces driven by functions asymptotic to $\kappa\sqrt{1-t}$. We prove a stability result when $\kappa \neq 4$ and show that $\kappa = 4$ can lead to non locally connected hulls. As a consequence, we obtain a driving term $\lambda(t)$ so that the hulls driven by $\kappa\lambda(t)$ are generated by a continuous curve for all $\kappa > 0$ with $\kappa \neq 4$ but not when $\kappa = 4$, so that the space of driving terms with continuous traces is not convex. As a byproduct, we obtain an explicit construction of the traces driven by $\kappa\sqrt{1-t}$ and a conceptual proof of the corresponding results of Kager, Nienhuis and Kadanoff.

Contents

1	Introduction and Results	2
2	Basics	3
2.1	Definitions and first properties	3
2.2	Renormalization on $[0,1)$	5
2.3	A time change	5
2.4	Table of Notation and Terminology	6
3	Self-similar curves	7
3.1	Collisions	7
3.2	Spirals	10
3.3	Tangential intersection	13
3.4	Comments	14
4	Convergence of traces and driving terms	15
4.1	Uniform convergence of traces	15
4.2	Uniform convergence of driving terms	17
5	The spiral	22
5.1	Driving terms of asymptotically self-similar curves	23
5.2	Examples of asymptotically self-similar curves	24
6	Collisions	28

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1 Introduction and Results

Let $\lambda(t)$ be continuous and real valued and let $g_t : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$ be the solution to the Loewner equation

$$\frac{d}{dt}g_t(z) = \frac{2}{g_t(z) - \lambda(t)} \quad , \quad g_0(z) = z \in \mathbb{H}, \quad (1.1)$$

where \mathbb{H} is the upper half-plane. It is an old problem to determine, in terms of λ , when K_t is a simple (Jordan) arc. The focus in this paper is on driving terms λ that generate arcs that are simple for “time” $t < t_0$ but potentially self-intersect at time t_0 . It was shown in [MR1] and [Li] that if λ is Hölder continuous with exponent $1/2$ and if $\|\lambda\|_{1/2} < 4$, then there is a simple curve γ with $\gamma[0, t] = K_t$ and $\gamma \setminus \gamma(0) \subset \mathbb{H}$. The norm 4 is sharp as the examples $\lambda(t) = \kappa\sqrt{1-t}$ show: Indeed, by [KNK], γ touches back on the real line if $\kappa \geq 4$ (hence the driving term $\lambda(t) = \kappa$ for $0 \leq t \leq t_0$ and $\lambda(t) = \kappa\sqrt{t_0+1-t}$ for $t_0 \leq t \leq t_0+1$ has a self-intersection in \mathbb{H} for t_0 sufficiently large). It was also shown in [MR1] that there is a λ with $\|\lambda\|_{1/2} < \infty$ such that K_1 spirals infinitely often around some disc, and hence is not locally connected. The starting point of this paper is the observation that from the conformal mapping point of view, the zero angle cusp at the tangential self-intersection for $\lambda(t) = 4\sqrt{1-t}$ is very similar to the infinitely spiraling prime end, and that this is reflected in the driving terms:

Theorem 1.1. *If γ is a sufficiently smooth infinite spiral of half-plane capacity T , or if γ has a tangential self-intersection, then its driving term λ satisfies*

$$\lim_{t \rightarrow T} \frac{|\lambda(T) - \lambda(t)|}{\sqrt{T-t}} = 4.$$

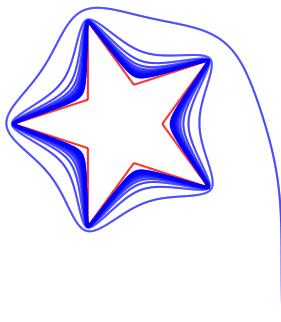


Figure 1: An infinite spiral converging towards a star.

See Sections 2.1 and 5 for the definitions and precise statements. In Section 5 we show that for every compact connected set $A \subset \mathbb{H}$ with connected complement, there is a sufficiently smooth infinite spiral winding infinitely often around A with limit set ∂A ; see Figure 1. The following natural question has been asked by Omer Angel: If the hull of λ is generated by a continuous curve γ and if $r < 1$, is it true that the hull of $r\lambda$ is generated by a continuous curve, too? In other words, is the space of driving terms of continuous curves starlike? We answer this question in the negative by proving

Theorem 1.2. *If γ is a sufficiently smooth infinite spiral of half-plane capacity T , and if λ is its driving term, then the trace of $r\lambda$ is continuous on the closed interval $[0, T]$ for all $r \neq \pm 1$.*

The main work is in proving a form of stability of the (nontangential) self-intersection of $\lambda(t) = \kappa\sqrt{1-t}$ for $\kappa > 4$:

Theorem 1.3. *If $\lambda : [0, T] \rightarrow \mathbb{R}$ is sufficiently regular on $[0, T)$ and if*

$$\lim_{t \rightarrow T} \frac{|\lambda(T) - \lambda(t)|}{\sqrt{T-t}} = \kappa > 4,$$

then

$$\gamma(T) = \lim_{t \rightarrow T} \gamma(t)$$

exists, is real and γ intersects \mathbb{R} in the same angle as the trace for $\kappa\sqrt{1-t}$.

See Section 6 for the statement of the necessary regularity. A similar result is true for $\kappa < 4$, see Theorem 6.2 in Section 6. By Theorems 1.3 and 6.2, the proof of Theorem 1.2 is reduced to proving sufficient regularity of the driving term of sufficiently smooth spirals. This is carried out in Proposition 5.9.

As mentioned above, the solutions to the Loewner equation driven by $\lambda(t) = \kappa\sqrt{1-t}$ were first computed in [KNK]. Their solutions are somewhat implicit and their analysis of the behaviour at the tip involved a little work. Our proof of Theorem 1.3 is based on the fact that the traces of $\lambda(t) = \kappa\sqrt{1-t}$ are fixed points of a certain renormalization operator, and that they take an extremely simple shape (they are straight lines and logarithmic spirals) after an appropriate change of coordinates. We therefore obtain an explicit “geometric construction” of the trace, which might be of independent interest. See Sections 2.2 and 3. We also need conditions and results about closeness of traces assuming closeness of driving terms, and vice versa. These are stated and proved in Sections 4.1 and 4.2.

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2 Basics

2.1 Definitions and first properties

In this section, we will fix some notation and terminology, as well as collect some standard properties. The expert can safely skip this section.

A *hull* is a bounded set $K \subset \mathbb{H}$ is such that $\mathbb{H} \setminus K$ is connected and simply connected. If g_K is a conformal map of $\mathbb{H} \setminus K$ onto \mathbb{H} such that $|g_K(z)| \rightarrow \infty$ as $z \rightarrow \infty$, let $\tilde{K} = \overline{K \cup K^{\mathbb{R}}} \cup_j I_j$, where $K^{\mathbb{R}}$ is the reflection of K about \mathbb{R} and $\{I_j\}$ are the bounded intervals in $\mathbb{R} \setminus \overline{K \cup K^{\mathbb{R}}}$. Then by the Schwarz reflection principle, g_K extends to be a conformal map of $\mathbb{C}^* \setminus \tilde{K}$ onto $\mathbb{C}^* \setminus I$ where \mathbb{C}^* is the extended plane and I is an interval contained in \mathbb{R} . Composing with a linear map $az + b$, $a > 0$, $b \in \mathbb{R}$, we may suppose that g_K has the hydrodynamic normalization

$$g_K(z) = z + \frac{2d}{z} + O\left(\frac{1}{z^2}\right) \tag{2.1}$$

near ∞ . If $f(z) \equiv g_K^{-1}(z) = z - 2d/z + \dots$ is continuous on $\overline{\mathbb{H}}$ then

$$f(z) - z = \int_I \frac{\operatorname{Im} f(x) dx}{x - z} \frac{1}{\pi}, \tag{2.2}$$

by the Cauchy integral formula or by the Poisson integral formula in \mathbb{H} applied to the bounded harmonic function $\operatorname{Im}(f(z) - z)$. Note that (2.2) implies that

$$2d = \lim_{z \rightarrow \infty} -z(f(z) - z) = \frac{1}{\pi} \int_I \operatorname{Im} f(x) dx > 0, \tag{2.3}$$

unless $f(z) \equiv z$. The coefficient d is called the half-plane capacity of K and is denoted by $d = \text{hcap}(K)$. It is easy to see that hcap is strictly increasing.

If $\lambda : [0, T] \rightarrow \mathbb{R}$ is continuous and $z \in \mathbb{H}$ then there are two cases for the solution $g_t(z)$ to the initial value problem (Loewner equation)

$$\frac{d}{dt}g_t(z) = \frac{2}{g_t(z) - \lambda(t)} \quad , \quad g_0(z) = z. \quad (2.4)$$

Either there is a time $T_z \leq T$ such that $\liminf_{t \rightarrow T_z} |g_t(z) - \lambda(t)| = 0$ (in this case it is not hard to show that $\lim_{t \rightarrow T_z} |g_t(z) - \lambda(t)| = 0$), or $\inf_{t \in [0, T]} |g_t(z) - \lambda(t)| > 0$. Set $T_z = \infty$ in the latter case. If

$$K_t = \{z \in \mathbb{H} : T_z \leq t\},$$

then $\mathbb{H} \setminus K_t$ is simply connected, and $g_t : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$ is the (unique) conformal map with $g_t(z) = z + 2t/z + O(1/z^2)$ near infinity. Thus each K_t is a hull and $\text{hcap}(K_t) = t$. We say that the hulls K_t are *driven by* λ and that λ is the *driving term for* K_t . We also say that K_t is *generated by a curve* γ if there is a continuous function $\gamma : [0, T] \rightarrow \overline{\mathbb{H}}$ such that for each $t \in [0, T]$, the domain $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. The curve γ is called the *trace* and we also say that g and γ are driven by λ and use the notation g^λ and γ^λ if necessary. It is known (see [MR2]) that the hulls driven by a sufficiently regular λ are simple (Jordan) curves, but that there are continuous λ whose hulls are not locally connected and hence not generated by a curve.

Consider a sequence of continuously growing hulls K_t with $K_0 = \emptyset$ (see [La] for a precise definition). Re-parametrizing K_t if necessary, we may assume that $\text{hcap}(K_t) = t$. Then the hydrodynamically normalized conformal maps $g_t \equiv g_{K_t} : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$ satisfy the Loewner equation for some continuous function $\lambda(t)$ and K_t are the hulls driven by λ . If g_t^{-1} has a continuous extension to $\lambda(t)$ then $g_t^{-1}(\lambda(t))$ is well-defined. If furthermore $\gamma(t) = g_t^{-1}(\lambda(t))$ is a continuous curve, then $K_t = \text{fill}(\gamma[0, t])$, where $\text{fill}(A)$ denotes the union of A and the bounded components of $\mathbb{H} \setminus A$, that is the complement of the unbounded component of $\mathbb{H} \setminus A$.

The standard example is provided by a continuous curve $\gamma \in \overline{\mathbb{H}}$, beginning in \mathbb{R} and without self-crossings but possibly self-touching, and $K_t = \text{fill}(\gamma[0, t])$. In this case, $g_t(\gamma(t)) = \lambda(t)$. Notice that in general, the trace $\gamma[0, t]$ is only a subset of the hull K_t , unless γ is a simple curve. For example, the hulls K_t on the middle left of Figure 2 are equal to the trace $\gamma[0, t]$ for all $t < 1$ (the κ in the figure is a parameter), but K_1 equals $\gamma[0, 1]$ together with the whole region enclosed by γ .

A crucial property is *scaling*: From

$$g_{rK}(z) = r g_K\left(\frac{z}{r}\right)$$

it follows that

$$\text{hcap}(rK) = r^2 \text{hcap}(K),$$

and that scaled hulls rK_t are driven by $r\lambda(t/r^2)$, if K is driven by λ . Since the function $\lambda(t) = \kappa\sqrt{t}$ is invariant under the scaling $\lambda \mapsto \frac{1}{r}\lambda(r^2t)$, it follows that its hulls are invariant under the geometric scaling $K \mapsto rK$. Notice that this would immediately imply that the hulls are rays $K_r = ar^2e^{i\theta}$ for some $a(K) > 0$, if we assume that K_r is generated by a simple curve. This of course also can be done by a direct computation.

Other crucial simple properties are the behaviour under *translation* (because $g_{K+x}(z) = g_K(z-x) + x$, the driving term of $\gamma+x$ is $\lambda+x$), under *concatenation* (if K_1 and K_2 are hulls driven by $\lambda_1 : [0, t_1] \rightarrow \mathbb{R}$ and $\lambda_2 : [0, t_2] \rightarrow \mathbb{R}$ and if $\lambda_1(t_1) = \lambda_2(0)$, then $K_1 * K_2 = K_1 \cup g_{K_1}^{-1}(K_2)$ is driven by $\lambda(t) = \lambda_1(t)1_{[0, t_1]} + \lambda_2(t-t_1)1_{(t_1, t_1+t_2]}$), and under *reflection* (if R_I denotes reflection in the imaginary axis, then $g_{R_I(K)} = R_I \circ g_K \circ R_I$ so that $R_I(K)$ is driven by $-\lambda$). We will often use the following version of the above concatenation: If $\gamma[0, t]$ is driven by λ , then $g_T(\gamma[T, t])$ is driven by $\tau \mapsto \lambda(T + \tau)$, for $0 \leq \tau \leq t - T$.

2.2 Renormalization on $[0,1]$

Let λ be continuous on $[0,1]$ and assume for ease of notation that the associated hulls K_t are generated by a curve $\gamma(t)$, $0 \leq t < 1$. In order to understand the trace γ (more generally the hulls K) near $t = 1$, we want to “pull down” the initial part $\gamma[0, T]$ of the curve by applying g_T , and then rescale the result so as to have half-plane capacity 1 again. For fixed $T \in [0, 1]$, the curve $\tilde{\gamma}_T = g_T(\gamma[T, 1])$ that is parametrized by

$$\tilde{\gamma}_T(t) = g_T(\gamma(T+t)), \quad 0 \leq t < 1-T \quad (2.5)$$

is driven by

$$\tilde{\lambda}_T(t) = \lambda(T+t), \quad 0 \leq t < 1-T. \quad (2.6)$$

Since $\tilde{\gamma}_T$ has capacity $1-T$, the scaled copy of $\tilde{\gamma}_T$

$$\gamma_T(t) \equiv \tilde{\gamma}_T(t(1-T))/\sqrt{1-T}, \quad 0 \leq t < 1 \quad (2.7)$$

has half-plane capacity 1. By Section 2.1, γ_T is the Loewner trace of

$$\lambda_T(t) = \lambda(T+t(1-T))/\sqrt{1-T}, \quad 0 \leq t < 1. \quad (2.8)$$

2.3 A time change

To facilitate our analysis of curves with driving term asymptotic to $\kappa\sqrt{1-t}$, we would like to reparametrize γ in a way that is well adapted to the renormalization operation (2.7). Let γ be a curve parametrized by half-plane capacity $t \in [0, 1]$. If $\gamma(T)$ and $\gamma(t)$ are consecutive points ($0 \leq T < t \leq 1$), then the renormalization of the arc between $\gamma(T)$ and $\gamma(t)$ has half-plane capacity $(t-T)/(1-T)$. In other words

$$\frac{g_T(\gamma(t))}{\sqrt{1-T}} = \gamma_T\left(\frac{t-T}{1-T}\right).$$

A parametrization $s(t)$ leaves “time-differences” invariant under renormalization provided

$$s(t) - s(T) = s\left(\frac{t-T}{1-T}\right) - s(0).$$

Dividing by $t-T$, passing to the limit $T \rightarrow t$ and integrating (after setting $s(0) = 0$ and $s'(0) = 1$), we therefore define

$$s = s(t) = \log \frac{1}{1-t}, \quad \text{or} \quad t = 1 - e^{-s}, \quad (2.9)$$

where $0 \leq s < \infty$. Set

$$G_s(z) = \frac{g_t(z)}{\sqrt{1-t}}, \quad F_s = G_s^{-1}, \quad \sigma(s) = \frac{\lambda(t)}{\sqrt{1-t}}, \quad \text{and} \quad \Gamma(s) = \gamma(t) \quad (2.10)$$

so that

$$G_s(\Gamma(s)) = \sigma(s) \quad \text{and} \quad F_s(\sigma(s)) = \Gamma(s)$$

We will say that Γ , G and F are *driven by* σ and write Γ^σ , G^σ and F^σ if necessary. By (2.10) and (1.1)

$$\dot{G}_s \equiv \frac{\partial}{\partial s} G_s = \frac{2}{G_s - \sigma(s)} + \frac{G_s}{2} \quad (2.11)$$

for all $z \in \mathbb{H} \setminus \Gamma[0, s]$, and

$$\frac{\dot{F}_s}{F'_s} = \frac{2}{\sigma - z} - \frac{z}{2} \quad (2.12)$$

for all $z \in \mathbb{H}$. This change of variables was used in [KNK] when $\lambda(t) = \kappa\sqrt{1-t}$, in which case $\sigma(s) \equiv \kappa$.

Convention: Throughout the remainder of the paper, the symbol s will refer to the “time change” defined by (2.9), whereas t will always stand for the parametrization by half-plane capacity.

We will now express the scaling relation (2.8) in terms of s and establish the semigroup property of G_s . The simple form may be the main advantage of the time change. Denote the shift of a function σ on $[0, \infty)$ by σ_u ,

$$\sigma_u(s) = \sigma(u + s) \quad \text{for } s \geq 0.$$

To simplify the notation, we set $\Gamma_{u,v} = G_u(\Gamma[u, v])$ and $\Gamma_u = \Gamma_{u,\infty}$. Then $\Gamma_{u,v}$ is the “pull-back” of the portion of Γ between “s-times” u and v , with initial point $\sigma(u) = \lambda(t(u)) \in \mathbb{R}$.

Lemma 2.1. *If the curve Γ is driven by σ , then the curve Γ_u is driven by σ_u . Moreover,*

$$G_{u+s}^\sigma = G_s^{\sigma_u} \circ G_u^\sigma \quad .$$

Proof. Fix u and set $\tau = 1 - e^{-u}$. By (2.5), (2.7), and (2.10), $G_u^\sigma(\Gamma[u, \infty])$ is the curve γ_τ . By (2.8) γ_τ is driven by λ_τ . Writing $t = 1 - e^{-s}$, we have

$$\frac{\lambda_\tau(t)}{\sqrt{1-t}} = \frac{\lambda(1 - e^{-u} + (1 - e^{-s})e^{-u})}{e^{-u/2}e^{-s/2}} = \frac{\lambda(1 - e^{-(u+s)})}{e^{-(u+s)/2}} = \sigma_u(s)$$

and hence $G_u^\sigma(\Gamma[u, \infty])$ is driven by σ_u . The semigroup property follows because both maps G_{u+s}^σ and $G_s^{\sigma_u} \circ G_u^\sigma$ are normalized conformal maps of the same domains, hence identical. \square

2.4 Table of Notation and Terminology

Notation	Brief definition
hull	bounded subset of \mathbb{H} with simply connected complement in \mathbb{H}
g_K	normalized conformal map $\mathbb{H} \setminus K$ onto \mathbb{H}
$\lambda(t)$	Loewner driving term
K_t	Loewner hull
γ	trace
$g_t \equiv g_{K_t}$	Loewner map from $\mathbb{H} \setminus K_t$ to \mathbb{H}
f_t	g_t^{-1}
$\tilde{\gamma}_T$	$g_T(\gamma[T, 1])$
γ_T	$g_T(\gamma[T, 1])/\sqrt{1-T}$
$\lambda_T(t)$	$\lambda(T + t(1-T))/\sqrt{1-T}$
s	$-\ln(1-t)$, and so $t = t(s) = 1 - e^{-s}$
$\Gamma(s)$	$\gamma(t(s))$
G_s	$e^{s/2}g_{1-e^{-s}}(z) = g_{t(s)}(z)/\sqrt{1-t(s)}$
F_s	G_s^{-1}
$\sigma(s)$	$e^{s/2}\lambda(1 - e^{-s}) = \lambda(t(s))/\sqrt{1-t(s)}$
$g_t^\lambda, \gamma^\lambda, G_s^\sigma, \Gamma^\sigma$	λ and σ are the corresponding driving terms
$\gamma^\kappa, \Gamma^\kappa$	traces with driving terms $\lambda(t) = \kappa\sqrt{1-t}$ and $\sigma(s) \equiv \kappa$, resp.
$\Gamma_{u,v}$	$G_u(\Gamma[u, v])$
Γ_u	$\Gamma_{u,\infty}$
$B^{\mathbb{R}}$	reflection of B about \mathbb{R}

3 Self-similar curves

We now describe the driving terms of curves γ for which $\tilde{\gamma}_T$ and γ are similar for each T . Here we call two subsets $A, B \subset \mathbb{H}$ *similar* if they differ only by a dilation and translation fixing \mathbb{H} . We say that γ is *self-similar* if $\tilde{\gamma}_T$ is similar to γ for every $0 < T < 1$.

We then give an explicit construction of such curves.

Proposition 3.1. *The curve γ is self-similar if and only if $\lambda(t) = C + \kappa\sqrt{1-t}$ for some constants C and κ . Moreover, in this case, the renormalized curves γ_T satisfy*

$$\gamma_T - \gamma_T(0) = \gamma - \gamma(0)$$

for each $0 < T < 1$, and the fixpoints of the map $\gamma \mapsto \gamma_T$ are precisely the Loewner traces of $\lambda(t) = \kappa\sqrt{1-t}$.

Proof. Suppose $\tilde{\gamma}_T = a(T)\gamma + b(T)$. Then by (2.6) and Section 2.1, $\tilde{\gamma}_T$ is driven by

$$\lambda(T+t) = a\lambda\left(\frac{t}{a^2}\right) + b \quad (3.1)$$

for $0 < t < 1 - T$ and $0 < t < a^2$. Since these intervals must be the same, $a = \sqrt{1-T}$. Setting $t = 0$ in (3.1) we obtain $b = \lambda(T) - \lambda(0)\sqrt{1-T}$, and setting $t = 1 - T$ we obtain

$$\lambda(T) = \lambda(1) + (\lambda(0) - \lambda(1))\sqrt{1-T}$$

as desired. Conversely, if $\lambda(t) = C + \kappa\sqrt{1-t}$, then $\lambda_T(t) = C/\sqrt{1-T} + \kappa\sqrt{1-t}$ by (2.8) and so γ_T is a translate of γ for each T . Moreover $\gamma = \gamma_T$ if and only if $\lambda = \lambda_T$ if and only if $C = 0$. \square

Next we will construct curves γ which are invariant under renormalization up to translation, hence obtaining the traces of $\kappa\sqrt{1-t}$ for some values of κ . This approach has the advantage of being conceptual and simple, but the disadvantage that it does not yield κ . Each construction will be followed by an explicit computation of the associated conformal maps, which then determines the associated constant κ .

3.1 Collisions

Fix θ with $0 < \theta < 1$. Let $D_\theta = \mathbb{H} \setminus S_\theta$ where S_θ is the line segment in \mathbb{H} from 0 to $e^{i\pi\theta}$. See the upper right corner of Figure 2. Let R_θ be the ray $\{e^{i\pi\theta}r : r \geq 1\}$ joining $e^{i\pi\theta}$ and ∞ in D_θ . Viewed as a ‘‘chordal Loewner trace’’ in D_θ from $e^{i\pi\theta}$ to ∞ , R_θ has the following similarity property: Parametrizing R_θ by $R_\theta(r) = e^{i\pi\theta}r$, the conformal map $z \mapsto z/r$ maps $D_\theta \setminus R_\theta[1, r]$ onto D_θ and maps $R_\theta[r, \infty)$ onto R_θ . If we transplant the map $z \mapsto z/r$ to \mathbb{H} by conjugating with a conformal map k of \mathbb{H} onto D_θ then we will obtain a self-similar (in the sense of Proposition 3.1) curve $\gamma = k^{-1}(R_\theta)$ provided ∞ is fixed. In other words, $k(\infty)$ must be fixed by the map $z \mapsto z/r$. If $k(\infty) = \infty$ then γ will be unbounded, and hence have infinite half-plane capacity. The only other choices for the image of ∞ are the two prime ends (boundary points) 0^+ and 0^- of D_θ at 0. Choose k so that $k(\infty) = 0^+$. Parametrize $\gamma = k^{-1}(R_\theta)$ by half-plane capacity, hcap, so that $\gamma(0) = k^{-1}(e^{i\theta})$ and $\gamma(1) = k^{-1}(\infty)$ (we may replace $k(z)$ by $k(cz)$ for some constant $c > 0$ so that $\text{hcap}(\gamma) = 1$). Suppose γ is driven by λ . If $r > 1$ is defined by $re^{i\pi\theta} = k(\gamma(T))$, then $G(z) = k^{-1}(\frac{1}{r}k(z))$ is a conformal map from $\mathbb{H} \setminus \gamma[0, T]$ to \mathbb{H} fixing ∞ and hence must equal $a(T)g_T + b(T)$ for some real constants a and b . Since $G(\gamma[T, 1]) = \gamma$, $\tilde{\gamma}_T$ is similar to γ by (2.5). Proposition 3.1 then guarantees $\lambda(t) = C + \kappa\sqrt{1-t}$. There is still one free (real) parameter in the definition of k , so we may assume that $k^{-1}(e^{i\theta}) = \kappa$ and thus $\gamma(0) = \lambda(0) = \kappa$

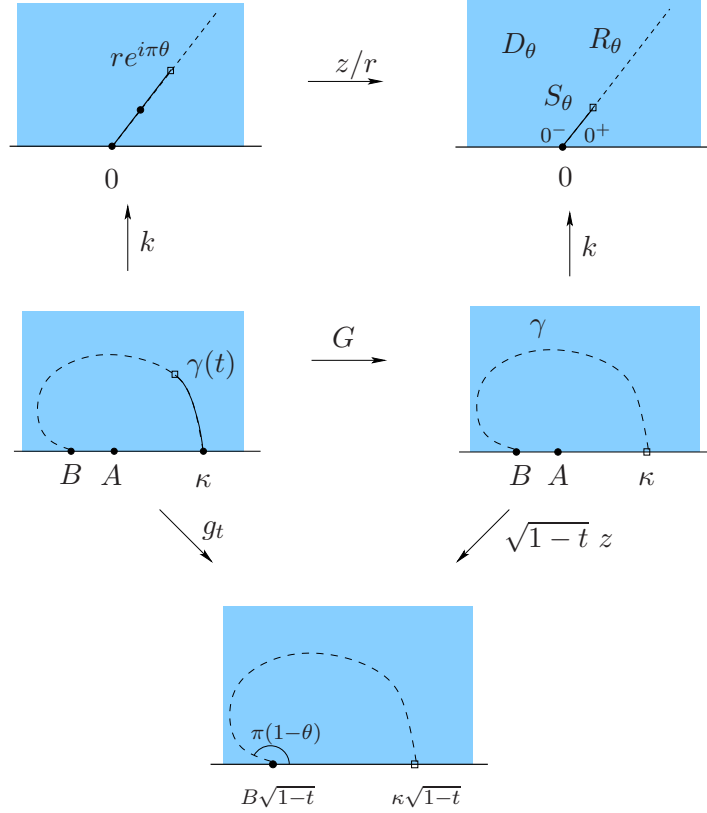


Figure 2: $\kappa > 4$:
 Loewner flow $z \mapsto z/r$ on the slit half-plane,
 time changed Loewner flow G on \mathbb{H} ,
 and Loewner flow g_t on \mathbb{H} .

and $\lambda(t) = \kappa\sqrt{1-t}$. Notice that γ “collides” with \mathbb{R} at $\gamma(1) = k^{-1}(\infty)$ forming an angle of $\pi(1-\theta)$ with the half line $[k^{-1}(\infty), +\infty)$.

To compute the relation between θ and κ , we will compute the corresponding conformal maps explicitly. The maps k are the fundamental building blocks for the numerical conformal mapping method called “zipper” [MR2]. By the Schwarz reflection principle or by Caratheodory’s theorem, k satisfies

$$\arg k(x) = \begin{cases} \pi\theta & \text{for } A < x \\ \pi & \text{for } B < x < A \\ 0 & \text{for } x < B. \end{cases}$$

where $k(A) = 0^-$ and $k(B) = \infty$. By Lindelöf’s maximum principle [GM, page 2],

$$\arg k(z) = \pi\theta + (1-\theta)\arg(z-A) - \arg(z-B)$$

and so

$$k(z) = ce^{i\pi\theta} \frac{(z-A)^{1-\theta}}{z-B} \quad (3.2)$$

where c is a positive constant chosen so that the length of S_θ will be equal to 1. In fact for any choice $B < A$ and appropriate c , the right side of (3.2) will be a one-to-one analytic map of \mathbb{H} onto D_θ because it is the composition of k with a linear map. Set

$$G(z) = G_s(z) = k^{-1}\left(\frac{1}{r}k(z)\right), \quad (3.3)$$

where $r = r(s)$ will be determined shortly. Then

$$\dot{G} = -\frac{\dot{r}}{r^2} \frac{k}{k' \circ G} = -\frac{\dot{r}}{r} \frac{k}{k'} \circ G.$$

Computing k'/k from (3.2) and simplifying we obtain

$$\dot{G} = \frac{\dot{r}}{r} \frac{(G-A)(G-B)}{(\theta G + (1-\theta)B - A)}.$$

Set $\dot{r}/r = \theta/2$, $AB = 4$ and $A + B = (A - (1-\theta)B)/\theta$. Then (2.11) holds with constant $\sigma \equiv A + B$, and hence (1.1) holds with

$$g_t(z) = \sqrt{1-t} G_{s(t)}$$

and $\lambda(t) = (A+B)\sqrt{1-t}$. We can now compute the relation between $\kappa = A+B$ and θ : If $A > 0$ then since $AB = 4$ and $A+B = (A - (1-\theta)B)/\theta$,

$$A = \frac{2}{\sqrt{1-\theta}} \quad \text{and} \quad B = 2\sqrt{1-\theta} \quad (3.4)$$

and

$$\kappa = A+B = 2\sqrt{1-\theta} + \frac{2}{\sqrt{1-\theta}}. \quad (3.5)$$

We also deduce that $r(s) = e^{s\theta/2} = (1-t)^{-\theta/2}$, and $c = 2^\theta(1-\theta)^{\theta/2-1}$ since $k(\kappa) = e^{i\pi\theta}$. The trace γ is a curve beginning at κ which ‘‘collides’’ with \mathbb{R} at $B = 2\sqrt{1-\theta}$ forming an angle of $\pi(1-\theta)$ with $[B, \infty)$. Note that the interval $0 < \theta < 1$ corresponds to the interval $4 < \kappa < \infty$. To obtain the maps for $-\infty < \kappa < -4$, simply reflect the construction above about the imaginary axis.

In summary, we conclude:

Proposition 3.2. *Given $\kappa > 4$, set $\theta = 2(1 + \kappa/\sqrt{\kappa^2 - 16})^{-1}$ and*

$$k(z) = e^{i\pi\theta} \frac{(z - 2/\sqrt{1-\theta})^{1-\theta}}{z - 2\sqrt{1-\theta}}$$

and

$$g_t(z) = (1-t)^{\frac{1}{2}} k^{-1}\left((1-t)^{\frac{\theta}{2}} k(z)\right)$$

Then k is a conformal map of \mathbb{H} onto $\mathbb{H} \setminus S_\theta$ where S_θ is a line segment in \mathbb{H} beginning at 0 and forming an angle $\pi\theta$ with $[0, \infty)$, and g_t satisfies the Loewner equation

$$\dot{g}_t = \frac{2}{g_t - \kappa\sqrt{1-t}},$$

with $g_0(z) \equiv z$. The trace $\gamma = k^{-1}(\{re^{i\pi\theta} : r > 0\} \setminus S_\theta)$ is a curve in \mathbb{H} which meets \mathbb{R} at angle $\frac{\pi}{2}$ at $\gamma(0) = \kappa$ and at angle $\pi(1-\theta)$ at $\gamma(1) = 2\sqrt{1-\theta}$. The case $\kappa < -4$ can be obtained from the case $\kappa > 4$ by reflecting about the imaginary axis.

In the statement of Proposition 3.2 we have replaced c (from (3.2)) by 1 for simplicity. Indeed the definition of γ and G do not depend on the choice of c . Changing c only changes the size of the slit S_θ . Here the length of the slit is $|k(z_0)|$, where z_0 is the solution of $k'(z_0) = 0$.

3.2 Spirals

Another type of region with a self-similarity property is a logarithmic spiral. Fix $\theta \in (0, \pi/2)$, set $\zeta = e^{i\theta}$ and consider the logarithmic spiral

$$S_\theta(t) = e^{t\zeta}, \quad -\infty \leq t \leq \infty. \quad (3.6)$$

Set $S^1 = S_\theta[0, \infty)$, $D_\theta = \mathbb{C} \setminus S^1$ and let R_θ be the curve $R_\theta(t) = S_\theta(-t)$, $t \geq 0$. See the upper right corner of Figure 3. Viewed as a ‘‘Loewner trace’’ in D_θ from the boundary point 1 of D_θ to the interior point 0, R_θ has the following self-similarity property: $z \mapsto e^{t\zeta}z$ maps $D_\theta \setminus R_\theta[0, t]$ onto D_θ and $R_\theta[t, \infty)$ onto R_θ . As before, it follows that any conformal map $k : \mathbb{H} \rightarrow D_\theta$ that fixes ∞ sends R_θ to a curve $\gamma \subset \mathbb{H}$ driven by $\lambda(t) = a + \kappa\sqrt{\text{hcap}(\gamma) - t}$. Notice that now the endpoint of γ is an interior point of \mathbb{H} , and because conformal maps are asymptotically linear, we see that γ is asymptotically similar to the logarithmic spiral at the endpoint.

To compute the relation between θ and κ , we will compute the corresponding conformal maps explicitly as in the case $\kappa > 4$. However, more work is required because Lindelöf’s maximum principle applies only to bounded harmonic functions, which we do not have in this case. Let $\beta = k^{-1}(0) \in \mathbb{H}$ and let $\gamma_0 = k^{-1}(-S_\theta)$. Then γ is a Jordan arc in \mathbb{H} from $k^{-1}(1)$ to β , and γ_0 is an arc in $\mathbb{H} \setminus \gamma$ from β to ∞ . See Figure 3. We can define a single-valued branch of $\log k(z)$ in $\mathbb{H} \setminus \gamma_0$, with $\log k(k^{-1}(1)) = 0$, so that for $z \in \mathbb{R}$

$$\log k(z) \in e^{i\theta}\mathbb{R}^+$$

and so that for $z \in \gamma = k^{-1}(R_\theta)$

$$\log k(z) \in -e^{i\theta}\mathbb{R}^+,$$

where $\mathbb{R}^+ = \{x > 0\}$. Note that for $x \in \mathbb{R}$

$$\overline{\log(x - \beta)} = \log(x - \bar{\beta})$$

for continuous branches of the logarithms in $\overline{\mathbb{H}} \setminus \gamma_0$, chosen so that $\lim_{x \rightarrow +\infty} \arg(x - \beta) = 0$, and $\lim_{x \rightarrow +\infty} \arg(x - \bar{\beta}) = 0$. The function

$$\log(z - \beta) + e^{2i\theta} \log(z - \bar{\beta}) = e^{i\theta} [e^{-i\theta} \log(z - \beta) + e^{i\theta} \log(z - \bar{\beta})]$$

then maps \mathbb{R} into the line with slope $\tan \theta$. This suggests the following candidate for k :

$$k_1(z) = (z - \beta)(z - \bar{\beta})^{e^{2i\theta}}, \quad (3.7)$$

which is analytic in \mathbb{H} , with $k_1(\mathbb{R}) \subset S_\theta$, just like k . Then

$$\frac{k'_1(x)}{k_1(x)} = e^{i\theta} \left(\frac{x(e^{i\theta} + e^{-i\theta}) - (\beta e^{i\theta} + \bar{\beta} e^{i\theta})}{(x - \beta)(x - \bar{\beta})} \right),$$

which points in the direction $e^{i\theta}$ for $x > \kappa$ and in the direction $-e^{i\theta}$ for $x < \kappa$, where κ is the zero of k'_1 . Thus as x varies from $-\infty$ to $+\infty$, $\log k_1$ traces a half line from ∞ to the tip $\log k_1(\kappa)$ and then back again. Let C be the boundary of a large half disk, given by the line segment from $-R$ to R followed by a semicircle in \mathbb{H} from R to $-R$. For large $|z|$, $k_1(z)$ is asymptotic to $k_2(z) = z^{1+e^{2i\theta}}$, and

$$\frac{\partial \arg k_2}{\partial \arg z} = 1 + \cos 2\theta > 0.$$

Thus $\arg k_2(z)$ increases as the semicircle is traced in the positive sense and the total change in $\arg k_2$ along the semi-circle is $\pi(1 + \cos 2\theta)$, which is at most 2π . Thus as z traces the curve

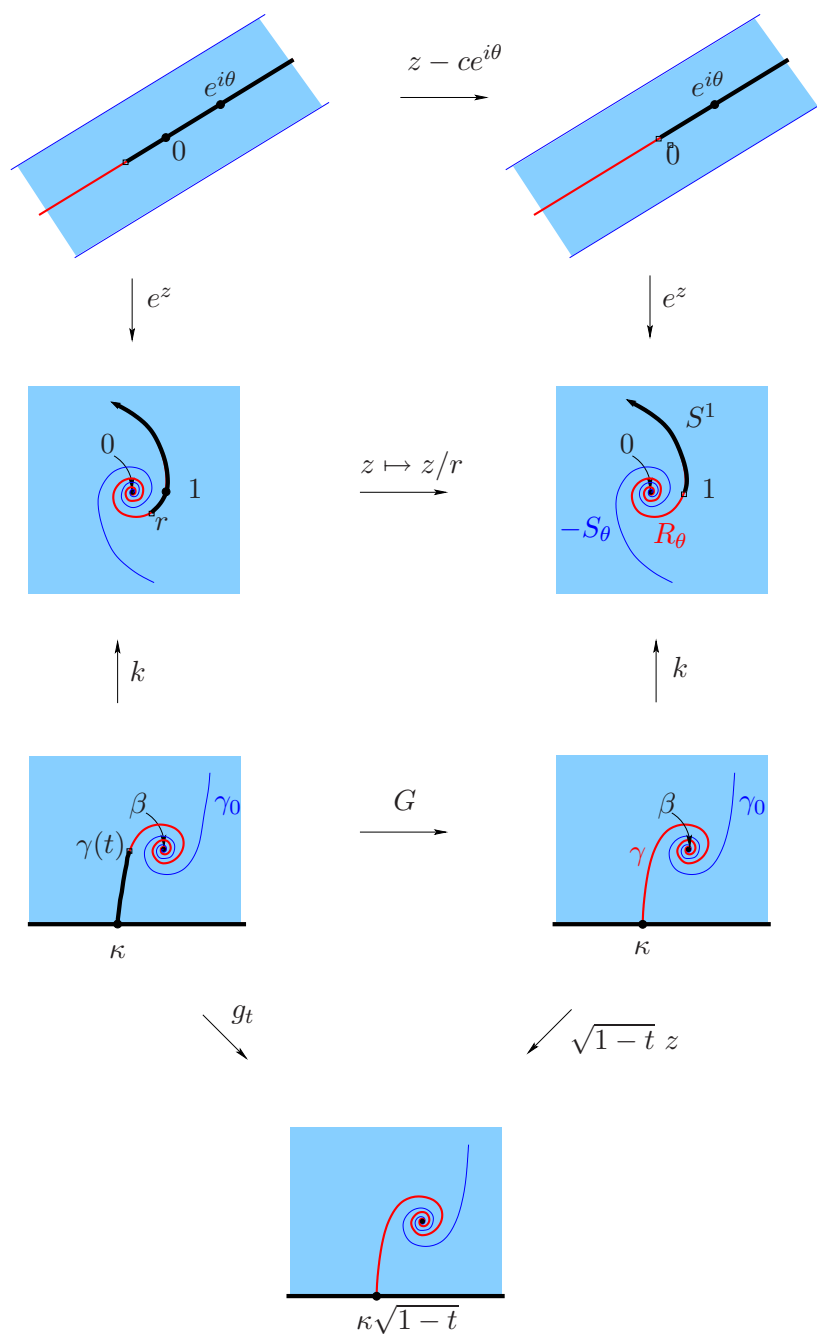


Figure 3: $0 < \kappa < 4$:
 Loewner flow $z \mapsto z/r$ on the complement of the spiral,
 time changed Loewner flow G on \mathbb{H} ,
 and Loewner flow g_t on \mathbb{H} .

C , k_1 traces a subarc of S_θ from $k_1(-R)$ to $k_1(\kappa)$ and back to $k_1(R)$, followed by a curve, on which $|z|$ is large, from $k_1(R)$ to $k_1(-R)$. By the argument principle, k_1 is a conformal map of \mathbb{H} onto $\mathbb{C} \setminus S^\kappa$ where S^κ is the subarc of S_θ from $k_1(\kappa)$ to ∞ .

It follows directly from the definition of S_θ that if $\zeta_1, \zeta_2 \in S_\theta$ then $\zeta_1/\zeta_2 \in S_\theta$, and so

$$k(z) = \frac{k_1(z)}{k_1(\kappa)}.$$

is a conformal map of \mathbb{H} onto $\mathbb{C} \setminus S^1$ such that $|k(z)| \rightarrow \infty$ as $z \in \mathbb{H} \rightarrow \infty$.

Define

$$G(z) = G_s(z) = k^{-1}\left(\frac{1}{r}k(z)\right) = k_1^{-1}\left(\frac{1}{r}k_1(z)\right), \quad (3.8)$$

where $r = r(s)$ will be determined shortly. Then

$$\dot{G} = -\frac{\dot{r}}{r^2} \frac{k_1}{k_1' \circ G} = -\frac{\dot{r}}{r} \frac{k_1}{k_1'} \circ G.$$

Computing k_1'/k_1 from (3.7) and simplifying we obtain

$$\dot{G} = -\frac{\dot{r}}{r} \frac{(G - \beta)(G - \bar{\beta})}{\left((1 + e^{2i\theta})G - (\bar{\beta} + \beta e^{2i\theta})\right)}.$$

Set $\dot{r}/r = -(1 + e^{2i\theta})/2$, with $r(0) = 1$, $|\beta| = 2$ and $\beta + \bar{\beta} = (\bar{\beta} + \beta e^{2i\theta})/(1 + e^{2i\theta})$. Then (2.11) holds with constant $\sigma \equiv \beta + \bar{\beta} = \kappa$, and hence (1.1) holds with

$$g_t(z) = \sqrt{1-t} G_{s(t)}$$

and $\lambda(t) = \kappa\sqrt{1-t}$. Note that $r(s) = e^{-s(\cos\theta)e^{i\theta}} \in R_\theta$ so that $z \mapsto rz$ maps S^1 to $S^1 \cup R_\theta[0, t]$ for some $t > 0$. Thus for each $r > 0$, G is analytic on $\mathbb{H} \setminus \gamma[0, t]$ for some $t > 0$ and maps $\mathbb{H} \setminus \gamma[0, t]$ onto \mathbb{H} .

We can now compute the relation between κ and θ : since $|\beta| = 2$ and $(\bar{\beta} + \beta e^{2i\theta})/(1 + e^{2i\theta}) = \beta + \bar{\beta}$, we conclude

$$\beta = 2ie^{i\theta}$$

and

$$\kappa = -4 \sin \theta.$$

The trace γ is a curve beginning at κ which spirals around $\beta \in \mathbb{H}$. Note that the interval $0 < \theta < \frac{\pi}{2}$ corresponds to the interval $-4 < \kappa < 0$. To obtain $0 < \kappa < 4$, we just reflect the construction about the imaginary axis; equivalently let $-\frac{\pi}{2} < \theta < 0$.

In summary, we conclude:

Proposition 3.3. *Given $0 < \kappa < 4$, set $\theta = -\sin^{-1}(\kappa/4)$, $\beta = 2ie^{i\theta}$ and*

$$k(z) = \frac{(z - \beta)(z - \bar{\beta})e^{2i\theta}}{(\kappa - \beta)(\kappa - \bar{\beta})e^{2i\theta}}$$

and

$$g_t(z) = (1-t)^{\frac{1}{2}} k^{-1}\left((1-t)^{-\cos\theta e^{i\theta}} k(z)\right).$$

Then k is a conformal map of \mathbb{H} onto $\mathbb{C} \setminus S^1$ where $S^1 = \{e^{te^{i\theta}} : t \geq 0\}$ is a logarithmic spiral in \mathbb{C} beginning at 1 and tending to ∞ and g_t satisfies the Loewner equation

$$\dot{g}_t = \frac{2}{g_t - \kappa\sqrt{1-t}},$$

with $g_0(z) \equiv z$. The trace $\gamma = k^{-1}(\{e^{-te^{i\theta}} : t > 0\})$ is a curve in \mathbb{H} beginning at $\kappa \in \mathbb{R}$ and spiraling around $\beta \in \mathbb{H}$. The case $-4 < \kappa < 0$ can be obtained from the case $4 > \kappa > 0$ by reflecting about the imaginary axis.

3.3 Tangential intersection

Now let D_0 be the domain $\mathbb{H} \setminus \{x + \pi i : x \leq 0\}$ and let R_0 be the halfline $\{x + \pi i : x \geq 0\}$, see Figure 4. Let $k : \mathbb{H} \rightarrow D_0$ be a conformal map normalized by $k(\infty) = p_{-\infty}$, where $p_{-\infty}$ is the prime end $\lim_{x \rightarrow -\infty} x + \pi i/2$. Then $\gamma = k^{-1}(R_0)$ has the self-similarity property: translation $z \mapsto z - r$ maps $D_0 \setminus [\pi i, \pi i + r]$ onto D_0 fixing $p_{-\infty}$. In this case γ intersects \mathbb{R} at $t = \text{hcap}(\gamma)$ tangentially.

Next we compute the conformal maps explicitly to show this case corresponds to $\kappa = 4$, and $\kappa = -4$ corresponds to the reflection of D_0 about the imaginary axis. These cases can also be obtained as limits of the collision case as $\theta \rightarrow 0$ or $\theta \rightarrow 1$, or as limits of the spiral case as $\theta \rightarrow -\frac{\pi}{2}$ or as $\theta \rightarrow \frac{\pi}{2}$.

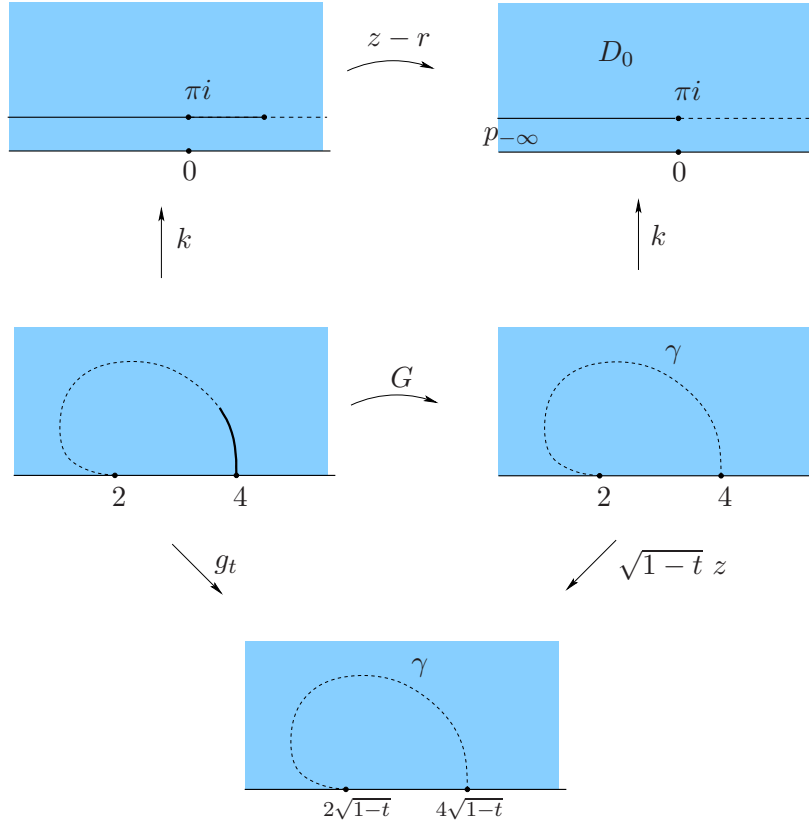


Figure 4: $\kappa = 4$:
 Loewner flow $z \mapsto z - r$ on the slit half-plane,
 time changed Loewner flow G on \mathbb{H} ,
 and Loewner flow g_t on \mathbb{H} .

As the case $|\kappa| < 4$, we will construct the map $k : \mathbb{H} \rightarrow D_0$. It is not enough to just construct an analytic function with the same imaginary part, as was done with the logarithm in the case $\kappa > 4$, since k is not bounded. Indeed, the identity function z has zero imaginary part on \mathbb{R} yet is nonconstant. It is perhaps easier to first construct a map k_1 of \mathbb{H} onto D_0 which maps ∞ to

∞ . Then $k_1(z) - \log(z)$ will have no jump in the imaginary part near ∞ , and so it must behave like $cz + d$ for z near ∞ , by the Schwarz reflection principle. Indeed the function defined by

$$k_1(z) = z + 1 + \log(z) \quad (3.9)$$

is analytic in \mathbb{H} and analytic across $\mathbb{R} \setminus \{0\}$. By calculus, k_1 is increasing on $(-\infty, -1)$, decreasing on $(-1, 0)$ and increasing on $(0, \infty)$ with $k_1(-1) = \pi i$. The imaginary part of k_1 is zero on $(0, \infty)$ and equal to π on $(-\infty, 0)$. Thus the image of \mathbb{R} by k_1 is the boundary of D_0 . Applying the argument principle to regions of the form $\mathbb{H} \cap \{r < |z| < R\}$ for small r and large R , we conclude that k_1 is a conformal map of \mathbb{H} onto D_0 . Also $k_1(0) = p_{-\infty}$ so that $k_1(-1/z)$ maps \mathbb{H} onto D_0 and sends ∞ to $p_{-\infty}$ as desired. Set

$$k(z) = k_1(-1/(Az + B)),$$

where $A > 0$ and $B \in \mathbb{R}$ and

$$G(z) = k^{-1}(k(z) - r) \quad (3.10)$$

where A , B , and $r(s)$ will be determined shortly. Then

$$\dot{G} = -\frac{\dot{r}}{k' \circ G} = \dot{r} \frac{(G + B/A)^2}{G + (B - 1)/A}$$

and if $B = -1$, $A = 1/2$, and $r = s/2$ then $\dot{r} = 1/2$ and (2.11) holds with $\sigma = 4$. Then the trace $\gamma = k^{-1}(R_0)$ is a curve in \mathbb{H} from $\kappa = 4$ to 2 which is tangential to \mathbb{R} at 2. In summary, we conclude

Proposition 3.4. *Let*

$$k(z) = \frac{4 - z}{2 - z} + \log\left(\frac{2}{2 - z}\right)$$

and

$$g_t(z) = (1 - t)^{\frac{1}{2}} k^{-1}\left(k(z) + \frac{1}{2} \log(1 - t)\right).$$

Then k is a conformal map of \mathbb{H} onto $\mathbb{H} \setminus \{x + \pi i : x \leq 0\}$ and g_t satisfies the Loewner equation

$$\dot{g}_t = \frac{2}{g_t - 4\sqrt{1 - t}},$$

with $g_0(z) \equiv z$. The trace $\gamma = k^{-1}(\{x + i : x > 0\})$ is a curve in \mathbb{H} that begins at 4, meeting \mathbb{R} at right angles, and ending at 2, where it is tangential to \mathbb{R} . The case $\kappa = -4$ can be obtained from the case $\kappa = 4$ by reflecting about the imaginary axis.

3.4 Comments

In [KNK], an implicit equation for g_t is found in each of the cases above. They find the explicit conformal maps only in the special case $\kappa = 3\sqrt{2}$ (see Section 5 of [KNK]). In this case $\gamma = \gamma_\kappa$ is a half circle and the example is closely related to the early work of Kufarev [K].

The maps g_t can also be computed without using Loewner's differential equation by simply normalizing the maps we have constructed at ∞ . For example, to determine A and B in the definition of k in (3.2) when $\kappa > 4$, we want

$$g_t(z) = \sqrt{1 - t} k^{-1}\left(\frac{1}{r}k(z)\right) = z + \frac{2t}{z} + \dots,$$

so that

$$\frac{1}{r}k(z) = k\left(\frac{g_t}{\sqrt{1 - t}}\right) = k \circ \left(\frac{z}{\sqrt{1 - t}} + \frac{2t}{z\sqrt{1 - t}} + \dots\right).$$

Thus

$$\frac{(1-t)^{-\theta/2}}{r} \frac{\left(1 - \frac{B\sqrt{1-t}}{z} + \frac{2t}{z^2} + O\left(\frac{1}{z^3}\right)\right)}{1 - \frac{B}{z}} = \left(\frac{1 - \frac{A\sqrt{1-t}}{z} + \frac{2t}{z^2} + O\left(\frac{1}{z^3}\right)}{1 - \frac{A}{z}}\right)^{1-\theta}.$$

Letting $z \rightarrow \infty$, we conclude that $r = (1-t)^{-\theta/2}$ and

$$1 + \frac{B(1 - \sqrt{1-t})}{z} + \frac{B^2(1 - \sqrt{1-t}) + 2t}{z^2} + O\left(\frac{1}{z^3}\right) = 1 + \frac{(1-\theta)A(1 - \sqrt{1-t})}{z} + \frac{1}{z^2} \left((1-\theta)(A^2(1 - \sqrt{1-t}) + 2t) + \frac{1}{2}(1-\theta)(-\theta)A^2(1 - \sqrt{1-t})^2 \right).$$

Equating coefficients, we obtain

$$B = (1-\theta)A \quad \text{and} \quad A^2 = \frac{4}{1-\theta},$$

which gives (3.4) as desired.

While it is possible to verify that g_t satisfies Loewner's equation directly from its definition and avoid the use of Section 3, the former approach using the renormalization was what led us to define k in the first place. The renormalization idea is of critical importance in Section 6.

If k_1 is the map (3.9) of \mathbb{H} to the half-plane minus a horizontal half line as in Section 3.3, then $-1/k_1(z)$ is a conformal map of the upper half plane to the upper half-plane minus a slit along a tangential circle. A careful analysis of the asymptotics of the driving term $\lambda(t)$, as $t \rightarrow 0$, for this curve was made in [PV] using the Schwarz-Christoffel representation. With the formula for the conformal map given here, an explicit expression for the driving term can be given.

4 Convergence of traces and driving terms

In this section we develop conditions under which a uniform estimate on the closeness of two driving terms $\|\lambda_1 - \lambda_2\|_\infty$ implies a uniform estimate on the corresponding traces, $\|\gamma_1 - \gamma_2\|_\infty$, and conditions under which a uniform estimate on the closeness of two traces implies a uniform estimate on the corresponding driving terms. Neither is true in general. An example of close driving terms whose traces are not close is described in page 116 of [La]. Figure 6 gives an example of close traces whose driving terms are not close. In Section 4.1 we will give a condition on a sequence of driving terms λ_n that guarantees uniform convergence of the traces γ_n , and in Section 4.2 we will give a geometric condition on traces that guarantees uniform convergence of their driving terms. These results are needed in Sections 6 and 5, respectively.

4.1 Uniform convergence of traces

If $\lambda_n \rightarrow \lambda$ and if additionally the sequence $\{\gamma_n\}$ is known to be equicontinuous, then uniform convergence follows easily. The following result makes use of this principle and applies to a large class of driving terms. Let $\lambda_1, \lambda_2 : [0, 1] \rightarrow \mathbb{R}$.

Theorem 4.1. *For every $\varepsilon > 0$, $C < 4$, and $D > 0$ there is $\delta > 0$ such that if*

$$\|\lambda_1 - \lambda_2\|_\infty < \delta \tag{4.1}$$

and if

$$|\lambda_j(t) - \lambda_j(t')| \leq C|t - t'|^{1/2} \tag{4.2}$$

whenever $|t - t'| < D$ and $j = 1$ or 2 , then the traces γ_1, γ_2 are Jordan arcs with

$$\sup_{t \in [0,1]} |\gamma_1(t) - \gamma_2(t)| < \varepsilon. \tag{4.3}$$

Proof. Analogous to the definition of quasislit discs in [MR1] we call a domain of the form $\mathbb{H} \setminus \gamma$ a K -quasislit half-plane if there is a K -quasiconformal map $F_\gamma : \mathbb{H} \rightarrow \mathbb{H}$ with $F_\gamma[0, i] = \gamma$. Thus $\mathbb{H} \setminus \gamma$ is a quasislit half-plane (for some K) if and only if γ is a quasiconformal arc in $\overline{\mathbb{H}}$ that meets \mathbb{R} non-tangentially. Equivalently $\gamma \cup \gamma^\mathbb{R}$ is a quasiconformal arc, where $\gamma^\mathbb{R}$ is the reflection of γ about \mathbb{R} . We will show first that $\mathbb{H} \setminus \gamma_1$ is a K -quasislit half-plane with K depending on C and D only. Let

$$\xi_j(t) = \lambda_1(t + jD), \quad 0 \leq j \leq \lfloor \frac{1}{D} \rfloor.$$

Then

$$g_t^{\lambda_1} = g_u^{\xi_N} \circ g_D^{\xi_{N-1}} \circ \dots \circ g_D^{\xi_1} \circ g_D^{\xi_0}$$

where

$$N = \lfloor \frac{t}{D} \rfloor \quad \text{and} \quad u = t - ND.$$

By assumption, $|\xi_j(t) - \xi_j(t')|^{\frac{1}{2}} \leq C|t - t'|$ for $t, t' \in [0, D]$ and it follows from ([Li], Theorem 2) that each $g_D^{\xi_j}(\mathbb{H})$ is a quasislit half-plane (with $K = K(C)$). It follows that $\gamma_1[0, 1]$ is the concatenation of $\lfloor \frac{1}{D} \rfloor + 1$ K -quasislit half-planes with $K = K(C)$. For the sake of completeness, we sketch a proof of the fact that the concatenation $\alpha * \beta$ of two quasislits α, β is a quasislit, see [MR1] for the disc version. Let $h_\alpha : \mathbb{H} \setminus \alpha \rightarrow \mathbb{H}$ be conformal and let $F_\beta : \mathbb{H} \rightarrow \mathbb{H}$ be K -quasiconformal with $F_\beta[0, i] = \beta$. Let $\psi(z) = \sqrt{z^2 + 1}$ be a normalized conformal map $\mathbb{H} \setminus [0, i] \rightarrow \mathbb{H}$.

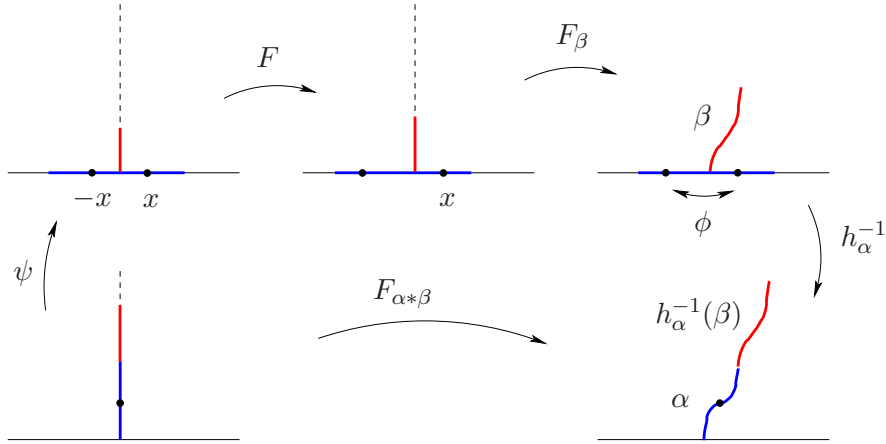


Figure 5: Concatenation $F_{\alpha * \beta} = h_\alpha^{-1} \circ F_\beta \circ F \circ \psi$.

We would like to find a qc map $F : \mathbb{H} \rightarrow \mathbb{H}$ with $F(i\mathbb{R}_+) = i\mathbb{R}_+$ so that $h_\alpha^{-1} \circ F_\beta \circ F \circ \psi$ is qc on \mathbb{H} . Thus we need $F_\beta(F(-x)) = \phi(F_\beta(F(x)))$ for $x \in [-1, 1]$, where $\phi : h_\alpha[\alpha] \rightarrow h_\alpha[\overline{\alpha}]$ is the (decreasing) welding homeomorphism, defined through $y = \phi(x) \Leftrightarrow h_\alpha^{-1}(x) = h_\alpha^{-1}(y)$. In order to construct such F , notice that ϕ has a quasisymmetric extension to \mathbb{R} by [MR1] and [Li]. It is easy to check that the function

$$F(x) = \begin{cases} x & \text{if } x \geq 0 \\ F_\beta^{-1}(\phi(F_\beta(-x))) & \text{if } x < 0 \end{cases}$$

is quasisymmetric. Extending F to \mathbb{H} in such a way that the imaginary axis is fixed (this can be done by using the Jerison-Kenig extension [JK], (see also [AIM], Chapter 5.8)), we have obtained the desired map F . Now $h_\alpha^{-1} \circ F_\beta \circ F \circ \psi(cz)$ is quasiconformal on $\mathbb{H} \setminus [0, i]$, for an

appropriate $c > 0$, and continuous on $[0, i]$, hence quasiconformal on \mathbb{H} . Thus $\alpha * \beta$ is a quasislit, and it follows by induction that γ_1 is a quasislit. The same argument applies to γ_2 .

Next, we claim that the parametrization of a K -quasislit by half-plane capacity has modulus of continuity depending on K only. Denote $g_t : \mathbb{H} \setminus \gamma[0, t] \rightarrow \mathbb{H}$ the normalized map, then $g_t(\gamma[t, t'])$ is a K -quasislit of capacity $t' - t$, and hence of diameter $\leq M\sqrt{|t' - t|}$ by [MR1, Lemma 2.5]. Because g_t^{-1} is Hölder continuous with bound depending on K only (by the John property of $\mathbb{H} \setminus \gamma$ and [P, Corollary 5.3]; see [W] for the modifications to \mathbb{H}), the claim follows.

To finish the proof of the theorem, let $\lambda_{n,1}$ and $\lambda_{n,2}$ satisfy (4.2) and $\|\lambda_{n,1} - \lambda_{n,2}\|_\infty \leq \frac{1}{n}$. Passing to a subsequence we may assume $\lambda_{n,j} \rightarrow \lambda$ uniformly. Denote γ the Loewner trace of λ . By the above equicontinuity of quasislits, we can pass to another subsequence and may assume that there are curves γ_j such that $\gamma_{n,j} \rightarrow \gamma_j$ uniformly. Now the Theorem follows from the next lemma. \square

Lemma 4.2. *If $\lambda_n \rightarrow \lambda$ and $\gamma_n \rightarrow \gamma_\infty$ uniformly, then $\gamma^\lambda = \gamma_\infty$. That is, γ_∞ is driven by λ .*

Proof. As $\gamma_n \rightarrow \gamma_\infty$, we have

$$\mathbb{H} \setminus \gamma_n[0, t] \rightarrow \mathbb{H} \setminus \gamma_\infty[0, t]$$

in the Caratheodory topology, for each t . Hence

$$f_n(t, z) \rightarrow f_{\gamma_\infty}(t, z)$$

uniformly on compact subsets of \mathbb{H} , and so

$$f'_n \rightarrow f'_{\gamma_\infty}$$

locally uniformly. Using

$$\dot{f}_n = f'_n \frac{2}{\lambda_n - z}$$

it follows that

$$\dot{f}_n \rightarrow f'_{\gamma_\infty} \frac{2}{\lambda - z}$$

for each z and each t , and that \dot{f}_n is uniformly bounded on $[0, T] \times \{z\}$ for each T and z . Thus

$$\begin{aligned} f_{\gamma_\infty}(t_1, z) - f_{\gamma_\infty}(t_2, z) &= \lim_{n \rightarrow \infty} (f_n(t_1, z) - f_n(t_2, z)) = \\ &= \lim_{n \rightarrow \infty} \int_{t_1}^{t_2} \dot{f}_n(t, z) dt = \int_{t_1}^{t_2} f'_{\gamma_\infty}(t, z) \frac{2}{\lambda(t) - z} dt \end{aligned}$$

by dominated convergence. Hence

$$\dot{f}_{\gamma_\infty} = f'_{\gamma_\infty} \frac{2}{\lambda - z}$$

and the lemma follows. \square

4.2 Uniform convergence of driving terms

In this section we develop geometric criteria for two hulls to have driving terms that are uniformly close. Figure 6 shows two hulls that are uniformly close but with large uniform distance between their driving terms. If $R_\theta = \{re^{i\theta} : 0 < r < 1\}$ and if $f(z) = \sqrt{z^2 - 4}$ then the image of R_ε by the map $f(z) - \varepsilon$ is similar to the left-hand curve in Figure 6 for small ε . The corresponding λ is equal to $-\varepsilon$ for $0 \leq t \leq 1$ and is $> -\varepsilon$ thereafter with a maximum value of approximately 1 by direct calculation (see Section 3.1). Likewise the image of $\mathbb{R}_{\pi-\varepsilon}$ by the map $f(z) + \varepsilon$ is similar to the right-hand curve in Figure 6. The corresponding λ is equal to ε for $0 \leq t \leq 1$ and is $< \varepsilon$

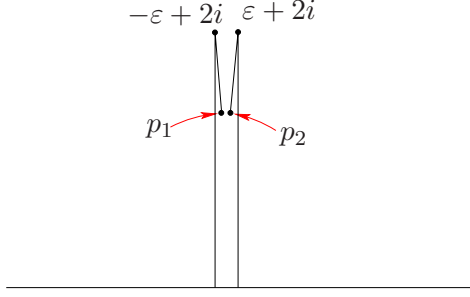


Figure 6: Close curves whose driving terms are not close.

thereafter with a minimum value of approximately -1 for small ε . The corresponding hulls γ_j satisfy

$$\sup |\gamma_1(t) - \gamma_2(t)| \leq 2\varepsilon,$$

but the driving terms satisfy

$$\sup |\lambda_1(t) - \lambda_2(t)| > 2 - 2\varepsilon.$$

Theorem 4.3. *Given $\varepsilon > 0$ and $c < \infty$, suppose A_1, A_2 are hulls with $\text{diam } A_j \leq 1$ and such that there there exists a hull $B \supset A_1 \cup A_2$ such that*

$$\text{dist}(\zeta, A_1) < \varepsilon \text{ and } \text{dist}(\zeta, A_2) < \varepsilon \text{ for all } \zeta \in \partial B.$$

Suppose further that there are curves $\sigma_j \subset \mathbb{H} \setminus A_j$ connecting a point $p \in \mathbb{H} \setminus B$ to $p_j \in A_j$ with $\text{diam } \sigma_j \leq c\varepsilon < \text{diam } A_j$, for $j = 1, 2$. If g_j is the hydrodynamically normalized conformal map of $\mathbb{H} \setminus A_j$ onto \mathbb{H} , for $j = 1, 2$, then

$$|g_1(p_1) - g_2(p_2)| \leq 2c_0\varepsilon^{\frac{1}{2}}(c^{\frac{1}{2}} + \rho),$$

where ρ is the hyperbolic distance from p to ∞ in $\Omega = \mathbb{C} \setminus \tilde{B}$, where $\tilde{B} = \overline{B \cup B^{\mathbb{R}}} \cup_j I_j$ and I_j are the bounded intervals in $\mathbb{R} \setminus \overline{B \cup B^{\mathbb{R}}}$.

For example, if the Hausdorff distance between $\overline{A_1}$ and A_2 is less than ε and if B is the complement of the unbounded component of $\mathbb{H} \setminus \overline{A_1 \cup A_2}$, then $\text{dist}(\zeta, A_j) < \varepsilon$ for all $\zeta \in B$, $j = 1, 2$. The theorem also applies in some situations where the Hausdorff distance between A_1 and A_2 is large. If p_1 and p_2 are the tips of the curves in Figure 6 then points p which are close to p_j have very large hyperbolic distance to ∞ .

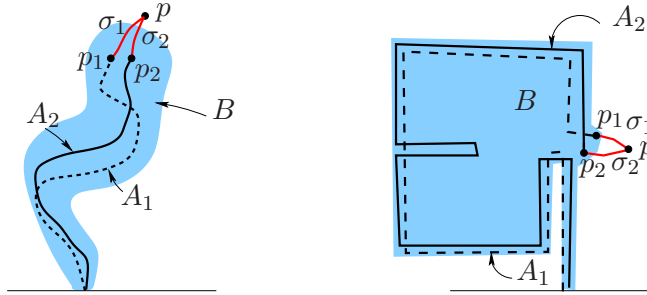


Figure 7: $g_1(p_1) - g_2(p_2)$ small.

The proof of Theorem 4.3 will follow from several lemmas. The first proposition is well known, but we include it for the convenience of the reader.

Proposition 4.4. *If A is a hull, let $\tilde{A} = \overline{A \cup A^{\mathbb{R}}} \cup_j I_j$ where $A^{\mathbb{R}}$ is the reflection of A about \mathbb{R} and $\{I_j\}$ are the bounded intervals in $\mathbb{R} \setminus \overline{A \cup A^{\mathbb{R}}}$. If g is the hydrodynamically normalized conformal map of the simply connected domain $\mathbb{C}^* \setminus \tilde{A}$ onto $\mathbb{C}^* \setminus I$ where I is an interval then*

$$\text{diam } A \leq \text{diam } I \leq 4 \text{diam } A. \quad (4.4)$$

Proof. Let G be the conformal map of $\Omega = \mathbb{C}^* \setminus \tilde{A}$ onto \mathbb{D} with $G(z) = a/z + b/z^2 + O(1/z^3)$ and $a > 0$. Then $g(z) = a(G + 1/G) + b/a$ so that

$$I = \left[-2a + \frac{b}{a}, 2a + \frac{b}{a}\right]$$

and $|I| = 4a$. Since $1/G(z)$ is a conformal map of Ω onto $\mathbb{C}^* \setminus \mathbb{D}$, we conclude that $a = \text{Cap}(\tilde{A})$, where $\text{Cap}(E)$ denotes the logarithmic capacity of E . If E is a connected set, $\text{Cap}(E)$ is decreased by projecting E onto a line, and increased if E is replaced by a ball containing E . The capacity of an interval is one-quarter of its length and the capacity of a ball is equal to its radius. Thus if E is connected, its capacity is comparable to its diameter and (4.4) follows. \square

The next lemma will be used to bound $|g_j(p_j) - g_j(p)|$.

Lemma 4.5. *There exist $c_0 < \infty$ so that if g is the hydrodynamically normalized conformal map of a simply connected domain $\mathbb{H} \setminus A$ onto \mathbb{H} and if S is a connected subset of $\mathbb{H} \setminus A$ then*

$$\text{diam } g(S) \leq c_0 \max(\text{diam } S, (\text{diam } A)^{\frac{1}{2}} (\text{diam } S)^{\frac{1}{2}}). \quad (4.5)$$

In particular if g is extended to be the conformal map of $\mathbb{C}^ \setminus \tilde{A}$ onto $\mathbb{C}^* \setminus I$ where I is an interval and $\tilde{A} = \overline{A \cup A^{\mathbb{R}}} \cup_j I_j$, where $A^{\mathbb{R}}$ is the reflection of A about \mathbb{R} and $\{I_j\}$ are the bounded intervals in $\mathbb{R} \setminus \overline{A \cup A^{\mathbb{R}}}$, then*

$$\text{dist}(g(z), I) \leq c_0 \max(\text{dist}(z, A), \text{dist}(z, A)^{\frac{1}{2}} \text{diam } A^{\frac{1}{2}}), \quad (4.6)$$

Proof. We will prove (4.6), then use it to prove (4.5). To prove (4.6) we may replace A , I , and $g(z)$ by cA , cI , and $cg(z/c)$, so that without loss of generality $|I| = 4$ and by (4.4) $1 \leq \text{diam } A \leq 4$. Fix $z = z_0 \in \mathbb{H}$. If $\text{dist}(z_0, A) \geq \text{diam } A$ then (4.6) follows from Koebe's estimate and the distortion theorem. (See [GM], Corollary I.4.4 and Theorem I.4.5). Suppose $\text{dist}(z_0, A) < \text{diam } A$ and let σ be a straight line segment from z_0 to A with $|\sigma| = \text{dist}(z_0, A)$. Set $\Omega = \mathbb{C}^* \setminus \tilde{A}$, set $\varphi(z) = |\sigma|/(z - z_0)$ and let $B = B(z_0, |\sigma|)$ be the ball centered at z_0 with radius $|\sigma|$. Then

$$\omega(\infty, \sigma, \Omega \setminus \sigma) \leq \omega(\infty, B, \Omega \setminus B) = \omega(0, \partial\mathbb{D}, \mathbb{D} \setminus \varphi(\tilde{A})).$$

The circular projection of $\varphi(\tilde{A})$ onto $[0, 1]$ is an interval $[|\sigma|/R, 1]$ where $R \geq \frac{1}{2} \text{diam } A$. By the Beurling projection theorem [GM], Theorem III.9.2, and an explicit computation, we obtain

$$\omega(\infty, \sigma, \Omega \setminus \sigma) \leq \omega(0, \partial\mathbb{D}, \mathbb{D} \setminus [|\sigma|/R, 1]) \leq \frac{4}{\pi} \tan^{-1}\left(\sqrt{\frac{|\sigma|}{R}}\right) \leq c_1 \sqrt{|\sigma|}. \quad (4.7)$$

Let G be the conformal map of Ω onto \mathbb{D} with $G(\infty) = 0$, with positive derivative at ∞ . Then by Beurling's projection theorem again,

$$\omega(\infty, \sigma, \Omega \setminus \sigma) = \omega(0, G(\sigma), \mathbb{D} \setminus G(\sigma)) \geq \omega(0, G(\sigma)^*, \mathbb{D} \setminus G(\sigma)^*)$$

where E^* is the circular projection of a set $E \subset \mathbb{D}$ onto $[0, 1]$. Again by an explicit computation

$$\omega(0, G(\sigma)^*, \mathbb{D} \setminus G(\sigma)^*) \geq \frac{1-r}{\pi} \quad (4.8)$$

where $r = \inf\{|z| : z \in G(\sigma)^*\} = \inf\{|z| : z \in G(\sigma)\}$. By Koebe's $\frac{1}{4}$ -theorem $r \geq r_0$, where r_0 does not depend on A . As in Proposition 4.4, $g = (G + 1/G) + b$, since $a = |I|/4 = 1$. Now if $w = G(z_0)$ and if ζ is the closest point in $\partial\mathbb{D}$ to w , set $x = \zeta + 1/\zeta + b \in I$. Then $|w - \zeta| \leq 1 - r$ and

$$|g(z_0) - x| = |w + 1/w - (\zeta + 1/\zeta)| = |w - \zeta| \left| 1 - \frac{1}{w\zeta} \right| \leq (1 - r) \left(1 + \frac{1}{r_0} \right). \quad (4.9)$$

Thus by (4.9), (4.8), and (4.7),

$$\text{dist}(g(z_0), I) \leq c\sqrt{|\sigma|} = c\sqrt{\text{dist}(z_0, A)}$$

proving (4.6).

To prove (4.5), if $\text{dist}(S, \partial\Omega) \geq \text{diam } S$ then for $z \in S$ by the Koebe distortion estimate and (4.6)

$$|g'(z)| \leq 4 \frac{\text{dist}(g(z), I)}{\text{dist}(z, A)} \leq c_0 \max\left(1, \left(\frac{\text{diam } A}{\text{dist}(z, A)}\right)^{\frac{1}{2}}\right).$$

If $z_1, z_2 \in S$, then by integrating g' along the line segment from z_1 to z_2 (which is contained in Ω) we obtain

$$|g(z_1) - g(z_2)| \leq c_0 \max(\text{diam } S, (\text{diam } A \text{ diam } S)^{\frac{1}{2}}).$$

If $\text{diam } S \geq \text{dist}(S, \partial\Omega)$ then we may rescale as in the proof of (4.6) so that $|I| = 4$ and $1 \leq \text{diam } A \leq 4$. Take $z_1 \in S$ so that $\text{dist}(z_1, \partial\Omega) \leq \text{diam } S$. Then $S \subset B = B(z_1, \text{diam } S)$ and $B \cap \partial\Omega \neq \emptyset$. As before, let $G : \Omega \rightarrow \mathbb{D}$ with $G(\infty) = 0$. By (4.7) and (4.8)

$$\frac{1 - r}{\pi} \leq \omega(\infty, B, \Omega \setminus B) \leq c_1 \sqrt{\text{diam } S},$$

where $r = \inf\{|z| : z \in G(B)\}$. If $G(B)_*$ denotes the *radial* projection of $G(B)$ onto $\partial\mathbb{D}$, then by Hall's lemma [D] and (4.7)

$$|G(B)_*| \leq 2\omega(0, G(B), \mathbb{D} \setminus G(B)) \leq c_2 \sqrt{\text{diam } S}.$$

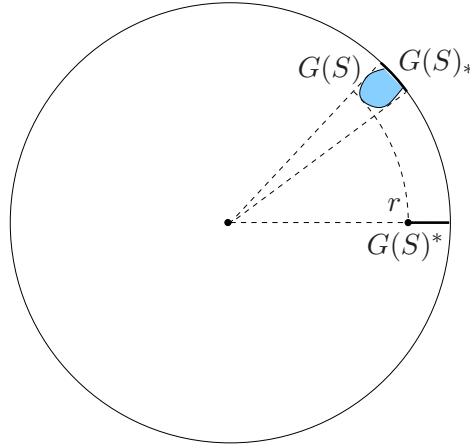


Figure 8: Diameter estimate via projections.

Since $G(S)$ is connected and $S \subset B$, we obtain

$$\text{diam } G(S) \leq c_3 \sqrt{\text{diam } S}.$$

If $r > 1/4$, this implies (4.5). If $r < 1/4$ then $\text{diam } S > \text{diam } A \geq 1$ and by (4.6)

$$\text{diam } g(S) \leq 2 \sup_{z \in S} \text{dist}(g(z), I) + |I| \quad (4.10)$$

$$\leq c_0 \sup_{z \in S} (\text{dist}(z, A), \text{dist}(z, A)^{\frac{1}{2}}) + 4 \leq c_4 \text{diam } S. \quad (4.11)$$

This proves (4.5). \square

Lemma 4.6. *If $|z| > 1$ then*

$$\int_{-1}^1 \frac{dt}{|t - \frac{1}{2}(z + \frac{1}{z})|} = 2 \log \frac{|z| + 1}{|z| - 1} = 2\rho_{\mathbb{C}^* \setminus \mathbb{D}}(z, \infty),$$

where ρ_{Ω} denotes the hyperbolic distance in Ω .

Proof. The integral can be computed explicitly and then simplified using $(z^{\frac{1}{2}} \pm z^{-\frac{1}{2}})^2 = z + 1/z \pm 2$. \square

The next lemma follows immediately from Lemma 4.6 and the conformal invariance of the hyperbolic metric.

Corollary 4.7. *If $I \subset \mathbb{R}$ is an interval and $\rho_{\mathbb{C}^* \setminus I}$ is the hyperbolic distance in $\mathbb{C}^* \setminus I$ then*

$$\int_I \frac{dt}{|t - z|} = 2\rho_{\mathbb{C}^* \setminus I}(z, \infty).$$

Lemma 4.8. *Suppose $A \subset B$ are hulls such that $\text{dist}(\zeta, A) < \varepsilon < 1$ for all $\zeta \in \partial B$. Let g_A and g_B be the hydrodynamically normalized conformal maps of $\mathbb{H} \setminus A$ and $\mathbb{H} \setminus B$ onto \mathbb{H} and let ρ be the hyperbolic distance from z to ∞ in $\Omega = \mathbb{C}^* \setminus \tilde{B}$ where $\tilde{B} = \overline{B \cup B^{\mathbb{R}}} \cup_j I_j$ and $B^{\mathbb{R}}$ is the reflection of B about \mathbb{R} and $\{I_j\}$ are the bounded intervals in $\mathbb{R} \setminus \overline{B \cup B^{\mathbb{R}}}$. Then for $z \in \mathbb{H} \setminus B$ and $0 < \varepsilon < \text{diam } A$*

$$|g_A(z) - g_B(z)| \leq c_0 (\text{diam } A)^{\frac{1}{2}} \rho \varepsilon^{\frac{1}{2}}$$

where c_0 is the constant in (4.6).

Proof. By (4.6), for $z \in \partial B$

$$\text{Im } g_A(z) \leq c_0 (\text{diam } A)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}.$$

Let $I_B \subset \mathbb{R}$ denote the interval corresponding to $g_B(B)$ and set $w = g_B(z)$, for $z \in \mathbb{H} \setminus B$. Then by (2.2) applied to $f = g_A \circ g_B^{-1}$

$$|g_A(z) - g_B(z)| = |g_A \circ g_B^{-1}(w) - w| = \left| \frac{1}{\pi} \int_{I_B} \frac{\text{Im } f(x)}{x - w} dx \right| \leq c_0 (\text{diam } A)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \frac{1}{\pi} \int_{I_B} \frac{dx}{|x - w|}.$$

By Corollary 4.7 and the conformal invariance of the hyperbolic metric

$$|g_A(z) - g_B(z)| \leq \frac{2c_0}{\pi} (\text{diam } A)^{\frac{1}{2}} \rho \varepsilon^{\frac{1}{2}}.$$

\square

Proof of Theorem 4.3. Let g_B be the hydrodynamically normalized conformal map of $\mathbb{H} \setminus B$ onto \mathbb{H} . Then

$$|g_1(p_1) - g_2(p_2)| \leq |g_1(p_1) - g_1(p)| + |g_1(p) - g_B(p)| + |g_B(p) - g_2(p)| + |g_2(p) - g_2(p_2)|$$

The desired inequality for the first and last terms follows from (4.5) since $\text{diam } \sigma_j \leq c\varepsilon < \text{diam } A_j$. The inequality for the second and third terms follows from Lemma 4.8. \square

In some circumstances it is preferable to use the hyperbolic metric in $\Omega_1 = \mathbb{C}^* \setminus \tilde{A}_1$ instead of $\Omega = \mathbb{C}^* \setminus \tilde{B}$. The next lemma says that if B is sufficiently close to A_1 , then we can do so.

Lemma 4.9. *If $\rho_{\Omega_1}(\infty, B) \geq c + \rho_{\Omega_1}(\infty, z)$ for some $c > 0$, then*

$$\rho_{\Omega}(\infty, z) \leq \rho_{\Omega_1}(\infty, z) + \log \frac{1}{1 - e^{-c}}.$$

Proof. Transfer the metric on Ω_1 to the disk and use the explicit form for the metric there. \square

We would like to end this section by considering this question: if two curves are close together, were they generated in approximately the same amount of time? In other words, are their half-plane capacities close? The lemma below addresses this.

Lemma 4.10. *Suppose A_1, A_2 are hulls with $\text{diam } A_j \leq 1$. Suppose there exist a hull $B \supset A_1 \cup A_2$ such that*

$$\text{dist}(\zeta, A_1) < \varepsilon \text{ and } \text{dist}(\zeta, A_2) < \varepsilon \text{ for all } \zeta \in B.$$

Then

$$|\text{hcap } A_1 - \text{hcap } A_2| \leq \frac{4}{\pi} c_0 \varepsilon^{\frac{1}{2}}.$$

Proof. Set $t_3 = \text{hcap } B - \text{hcap } A_1 > 0$. By (4.6) $|\text{Im } g_{A_1}(z)| \leq c_0 \varepsilon^{\frac{1}{2}}$ for $z \in B$. By (2.3) applied to $f = g_{A_1} \circ g_B^{-1}$ we conclude

$$t_3 \leq \frac{2}{\pi} c_0 \varepsilon^{\frac{1}{2}}.$$

The same argument applies to $t_4 = \text{hcap } B - \text{hcap } A_2$, and thus

$$|\text{hcap } A_1 - \text{hcap } A_2| \leq \frac{4}{\pi} c_0 \varepsilon^{\frac{1}{2}}.$$

\square

5 The spiral

We have noticed in Proposition 3.1 that self-similar curves are driven by $\kappa\sqrt{1-t}$. We will first generalize this by proving that curves which are ‘‘asymptotically self-similar’’ have driving terms asymptotic to $\kappa\sqrt{1-t}$. Then we will show that certain spirals are asymptotically self-similar.

5.1 Driving terms of asymptotically self-similar curves

Let $\gamma^{(n)} : [0, 1) \rightarrow \overline{\mathbb{H}}$ and γ be Loewner traces parametrized by half-plane capacity, with driving terms $\lambda^{(n)}$ and λ . We say that $\gamma^{(n)}$ converges to γ in the Loewner topology and write $\gamma^{(n)} \rightsquigarrow \gamma$ if for each $0 < t < 1$ we have

$$\sup_{0 \leq \tau \leq t} |\lambda^{(n)}(\tau) - \lambda(\tau)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Fix $\kappa \in \mathbb{R}$ and let γ^κ be the self-similar curve constructed in Section 3, driven by $\lambda^\kappa(t) = \kappa\sqrt{1-t}$. We will first show that if $\gamma : [0, 1) \rightarrow \mathbb{H}$ is such that the renormalized curves γ_T , translated so as to start at κ , converge to γ^κ in the Loewner topology, then λ behaves like λ^κ near $t = 1$.

Proposition 5.1. *If $\gamma_T - \gamma_T(0) + \kappa \rightsquigarrow \gamma^\kappa$ as $T \rightarrow 1$, then λ has a continuous extension to $[0, 1]$, and*

$$\lim_{t \rightarrow 1} \frac{\lambda(t) - \lambda(1)}{\sqrt{1-t}} = \kappa.$$

Proof. Fix $a < 1$ and set $T_n = 1 - a^n$. By (2.8), $\gamma_T - \gamma_T(0) + \kappa$ is driven by

$$\phi_T(t) = \lambda_T(t) - \gamma_T(0) + \kappa = \frac{\lambda(T + t(1-T)) - \lambda(T)}{\sqrt{1-T}} + \kappa.$$

By assumption, given $\varepsilon > 0$, there is an $n_0 < \infty$ so that if $n \geq n_0$ and $T_n \leq t' \leq T_{n+1}$ then

$$\left| \frac{\lambda(t') - \lambda(T_n)}{\sqrt{1-T_n}} + \kappa - \kappa\sqrt{1-t} \right| < \varepsilon,$$

where $t' = T_n + t(1-T_n)$ for some $0 \leq t \leq 1-a$. Thus

$$|\lambda(t') - \lambda(T_n) + \kappa(1 - \sqrt{1-t})a^{\frac{n}{2}}| < \varepsilon a^{\frac{n}{2}}. \quad (5.1)$$

In particular if $t' = T_{n+1}$ then

$$|\lambda(T_{n+1}) - \lambda(T_n) + \kappa(1 - \sqrt{a})a^{\frac{n}{2}}| < \varepsilon a^{\frac{n}{2}},$$

so that for $m > n \geq n_0$, by addition of these inequalities,

$$|\lambda(T_m) - \lambda(T_n) + \kappa a^{\frac{n}{2}}(1 - a^{\frac{m-n}{2}})| < \varepsilon a^{\frac{n}{2}} \frac{1 - a^{\frac{m-n}{2}}}{1 - \sqrt{a}}.$$

Since $a^k \rightarrow 0$, as $k \rightarrow \infty$, this proves $\{\lambda(T_m)\}$ is Cauchy. Set $\lambda(1) = \lim \lambda(T_m)$. Then

$$|\lambda(T_n) - \lambda(1) - \kappa a^{\frac{n}{2}}| < \varepsilon \frac{a^{\frac{n}{2}}}{1 - \sqrt{a}}. \quad (5.2)$$

Adding (5.1) and (5.2) gives

$$|\lambda(t') - \lambda(1) - \kappa\sqrt{1-t}a^{\frac{n}{2}}| < \varepsilon a^{\frac{n}{2}} \frac{2}{1 - \sqrt{a}},$$

for $n \geq n_0$. Since $1 - t' = (1-t)(1-T_n) = (1-t)a^n$ we conclude

$$\left| \frac{\lambda(t') - \lambda(1)}{\sqrt{1-t'}} - \kappa \right| < \frac{2\varepsilon}{(1 - \sqrt{a})a^{\frac{1}{2}}},$$

for $T_n \leq t' \leq T_{n+1}$ and $n \geq n_0$. The Proposition follows by letting $\varepsilon \rightarrow 0$. \square

5.2 Examples of asymptotically self-similar curves

Next, we present a class of examples γ that satisfy the assumption of Proposition 5.1. In order to keep the proofs as short and simple as possible, we will not give the most general definition, but restrict ourselves to the discussion of two specific examples. However, in remarks during the proofs we will emphasize the assumptions that the proofs really depend upon, allowing the reader to formulate and verify details of general conditions.

We first consider an infinite spiral that accumulates towards a given connected compact set as in Figure 1. Consider the curve $\nu_0 \in \mathbb{D}$ given by

$$\nu_0(t) = te^{\frac{i}{t-1}}, 0 \leq t < 1. \quad (5.3)$$

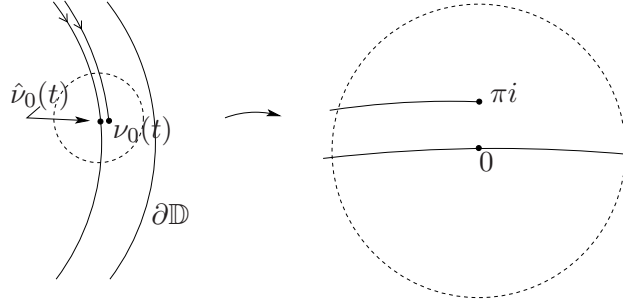


Figure 9: Disk spiral.

Remark 5.2. Denote $\hat{\nu}_0(t)$ the point on the “previous turn” with same argument as $\nu_0(t)$ (in formula: $\hat{\nu}_0(t) = \nu_0(\hat{t})$ where $\hat{t} = 1 + 1/(2\pi + 1/(t-1))$). Notice that the domain $\mathbb{D} \setminus \nu_0[0, t]$, translated by $\hat{\nu}_0(t)$ and dilated by $\pi i/(\nu_0(t) - \hat{\nu}_0(t))$, converges to the slit half-plane $D_0 = \mathbb{H} \setminus \{x + \pi i : x \leq 0\}$. See Figure 9.

Let $A \subset \mathbb{H}$ be compact such that $\mathbb{C} \setminus A$ is simply connected and let $f : \mathbb{D} \rightarrow \mathbb{C} \setminus A$ be a conformal map with $f(0) = \infty$. Replacing $f(z)$ by $f(e^{i\theta_0}z)$ we may choose t_0 so that $f(\nu_0(t_0)) \in \mathbb{R}$ and $f(\nu_0(t)) \in \mathbb{H}$ for $t > t_0$. Then $f(\nu_0(t))$, $t_0 \leq t < 1$, parametrizes a curve that begins in \mathbb{R} and winds around A infinitely often, accumulating at the outer boundary of A . For example, Figure 1 was created this way using the numerical conformal mapping routine “zipper” [MR1]. We will show that this curve satisfies Theorem 1.1 with $\kappa = 4$. To this end, scale this curve so that its half-plane capacity is 1 (that is, consider the curve $cf \circ \nu_0$ where $c^2 \text{hcap}(f(\nu_0[t_0, 1])) = 1$), and reparametrize by half-plane capacity. Call the resulting curve $\nu^A(t)$, and denote $\hat{\nu}^A(t)$ the point of the previous turn with same “argument” (formally, writing $\nu^A(t) = cf \circ \nu_0(u(t))$, we have $\hat{\nu}^A(t) = cf \circ \nu_0(\widehat{u(t)})$, where $\widehat{u(t)}$ is defined in Remark 5.2).

Theorem 5.3. *The curve ν^A satisfies the assumption of Proposition 5.1 with $\kappa = 4$, and consequently its driving term satisfies*

$$\lim_{t \rightarrow 1} \frac{\lambda(t) - \lambda(1)}{\sqrt{1-t}} = 4.$$

Recall the notation of Section 3.3, in particular the slit half-plane D_0 , the conformal map $k : \mathbb{H} \rightarrow D_0$ and the curve $\gamma = k^{-1}(\{x + \pi i : x \geq 0\})$, the trace of $4\sqrt{1-t}$. The key feature of $\nu = \nu^A$ (and therefore the curve ν_0 defined in (5.3)) is, roughly speaking, that $\mathbb{H} \setminus \nu[0, t]$ looks like D_0 when zooming in at $\nu(t)$. More precisely, we have

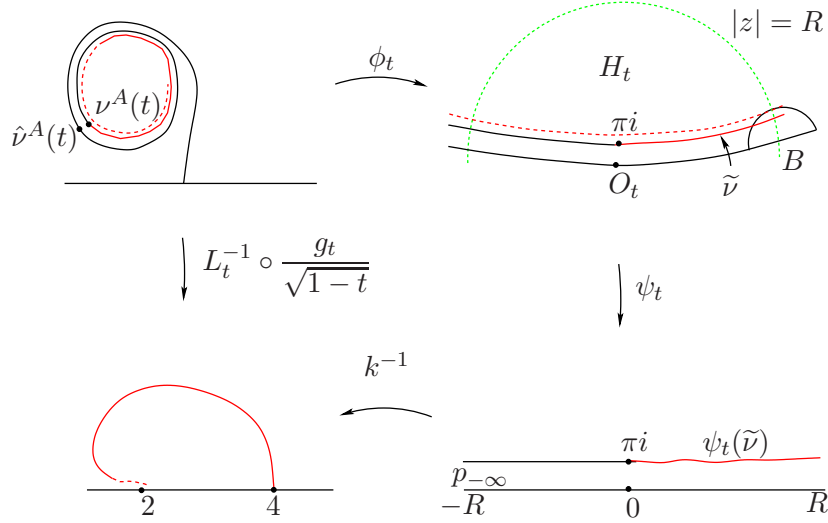


Figure 10: Decomposition of $g_t(z)$.

Lemma 5.4. *For each $t \in [0, 1)$, there is a linear map $\phi_t(z) = a_t z + b_t$ such that $\phi_t(\nu(t)) = \pi i$, and such that $\phi_t(\mathbb{H} \setminus \nu[0, t])$ converges to D_0 in the Caratheodory topology (with respect to the point $1 + \pi i$, say). Furthermore, $O_t = \phi_t(\hat{\nu}(t)) \rightarrow 0$ as $t \rightarrow 1$.*

Remark 5.5. Our curve ν_0 is sufficiently smooth so that Caratheodory convergence will be enough. For a general curve, we would need slightly stronger assumptions, see the remarks below.

Proof. This is an easy consequence of the Koebe distortion theorem and Remark 5.2, using that the distance $|\hat{\nu}_0(t) - \nu_0(t)|$ between consecutive turns is asymptotic to $2\pi(1-t)^2$ and therefore much smaller than the distance from $\nu_0(t)$ to $\partial\mathbb{D}$. \square

Next, let ψ_t denote the conformal map from $H_t = \phi_t(\mathbb{H} \setminus \nu[0, t])$ onto D_0 , normalized such that $\psi_t(\pi i) = \pi i$, $\psi_t(O_t) = 0$, and such that $\psi_t(\infty)$ equals the prime end $p_{-\infty} = k(\infty)$ (see Section 3.3).

Lemma 5.6. *For each $R > 0$ and $\epsilon > 0$ there is $t_0 < 1$ such that*

$$|\psi_t(z) - z| < \epsilon \quad \text{for all } z \in \tilde{\nu} \cup B \quad (5.4)$$

for $t > t_0$, where $\tilde{\nu}$ is the component of $\phi_t(\nu[t, 1]) \cap \{|z| < R\}$ containing πi , and B is the component of $H_t \cap \{|z - R| < 2\pi\}$ containing $R + \pi i$.

Proof. A standard application of Caratheodory convergence, provided by Lemma 5.4, requires normalization of conformal maps at an interior point (such as $\pi i + 1$). To deal with our situation, consider the conformal maps $\varphi_t : \mathbb{D} \rightarrow H_t$ and $\varphi_0 : \mathbb{D} \rightarrow D_0$, normalized by $\varphi_t(0) = \pi i + 1$ and $\varphi_t'(0) > 0$. By Lemma 5.4 we have $\varphi_t \rightarrow \varphi_0$ compactly as $t \rightarrow 1$. Denote a_t, b_t and c_t the preimages of ∞, O_t and πi under φ_t , and denote a, b, c the preimages of $p_{-\infty}, 0$ and πi under φ_0 . Set $T_t = \varphi_0^{-1} \circ \psi_t \circ \varphi_t$ so that T_t is the unique automorphism of \mathbb{D} that maps a_t, b_t, c_t to a, b, c .

It is not hard to see that $a_t \rightarrow a, b_t \rightarrow b$ and $c_t \rightarrow c$ as $t \rightarrow 1$: To prove $a_t \rightarrow a$, fix $\rho > 0$ large and consider the vertical line segment $A = (-\rho, -\rho + \pi i) \subset D_0$ and notice that $A' = \varphi_0^{-1}(A)$ is a crosscut of \mathbb{D} of small diameter separating 0 from a . The extremal distance from A to the

boundary arc of D_0 between 0 and πi containing ∞ (that is, the image under φ_0 of the subarc of $\partial\mathbb{D}$ between b and c) is large (it is of the order $e^{\rho\pi}$). By conformal invariance, the extremal distance between $\psi_t^{-1}(A)$ and the boundary arc between O_t and πi (that is, one turn of the spiral) is large, and it follows that the harmonic measure of $\psi_t^{-1}(A)$ at $\pi i + 1$ in H_t is small. In particular, there is $\rho' \leq \rho$ with $\rho' \rightarrow \infty$ as $\rho \rightarrow \infty$ ($\rho' = \rho/2$ will do) such that for $t \geq t_0(\rho)$ the component A_t of $(-\rho' + i\mathbb{R}) \cap H_t$ containing $-\rho' + i\pi/2$ separates $\pi i + 1$ and $\psi_t^{-1}(A)$ in H_t . Hence $\varphi_t^{-1}(A_t)$ separates 0 and a_t in \mathbb{D} . Denote $\alpha = \varphi_0^{-1}(-\rho - 1 + i\pi/2)$ so that α is contained in the component of $\mathbb{D} \setminus A'$ containing a . Because $\varphi_t(\alpha) \rightarrow \varphi_0(\alpha)$ as $t \rightarrow 1$, A_t separates $\pi i + 1$ and $\varphi_t(\alpha)$ in H_t . Consequently, α is also contained in the component of $\mathbb{D} \setminus \varphi_t^{-1}(A_t)$ containing a_t and we obtain $|a - a_t| \leq 2(\text{diam } A' + \text{diam } \varphi_t^{-1}(A_t))$ which can be made arbitrarily small by choosing ρ large and $t \geq t_0(\rho)$. The convergence $b_t \rightarrow b$ and $c_t \rightarrow c$ can be proved in a similar fashion, replacing A by small circular arcs centered at 0 and πi . We leave the details to the reader.

It follows that $T_t \rightarrow \text{id}$ uniformly in \mathbb{D} . Hence uniform convergence to 0 of $\psi_t(z) - z = \varphi_0(T_t(w)) - \varphi_t(w)$, writing $w = \varphi_t^{-1}(z)$, follows from the convergence $\varphi_t \rightarrow \varphi_0$ as long as z stays boundedly close to $\pi i + 1$ in the hyperbolic metrics of H_t . This proves (5.4) on $\tilde{\nu} \setminus \{|z - \pi i| < \delta\}$, for each $\delta > 0$. Because $\text{diam } \varphi_t^{-1}(\{|z - \pi i| < \delta\}) < C\sqrt{\delta}$, $c_t \in \varphi_t^{-1}(\{|z - \pi i| < \delta\})$, $c = T_t(c_t)$, and φ_0 is continuous near c , (5.4) also holds on $\tilde{\nu} \cap \{|z - \pi i| < \delta\}$ by choosing δ small. Finally, (5.4) on B follows by extending ψ_t^{-1} across the interval $[0, 2R]$ using Schwarz reflection, and noticing that $\{|z - R| < 2\pi\}$ is uniformly compactly contained in the extended domains H_t for t sufficiently large. \square

Remark 5.7. For more general curves, the validity of the conclusion of the previous lemma requires some mild regularity of ν in addition to the Caratheodory convergence of the rescaled domains H_t : Indeed, if $\phi_t(\nu(t)) = \pi i$ cannot be joined to $\pi i + 1$ within H_t by a curve of diameter close to 1, then ψ_t cannot be close to the identity near πi . Assuming for instance that the component of $H_t \cap D(0, 2\pi)$ with πi in its boundary is a John domain is enough to guarantee the conclusion of the lemma on $\tilde{\nu}$. Assuming that $\tilde{\nu}$ is a $K(t)$ -quasicircle with $K(t) \rightarrow 1$ as $t \rightarrow 1$ is enough to guarantee (5.4) on B .

Proof of Theorem 5.3. We need to show that the curves $\nu_T = g_T(\nu[T, 1])/\sqrt{1-T}$, translated so as to start at $\kappa = 4$, converge to $\gamma = \gamma^4$ in the Loewner topology as $T \rightarrow 1$. To see this, observe that $g_T/\sqrt{1-T} = L_T \circ k^{-1} \circ \psi_T \circ \phi_T$ for some linear self-map $L_T(z) = \alpha_T z + \beta_T$ of $\mathbb{H} : \mathbb{H}$. Indeed, the map $k^{-1} \circ \psi_T \circ \phi_T$ is a conformal map from $\mathbb{H} \setminus \nu[0, T]$ onto \mathbb{H} fixing ∞ .

Next, we claim that $\alpha_T \rightarrow 1$ as $T \rightarrow 1$. Take R large and consider the component $\tilde{\nu} = \tilde{\nu}(T, R)$ of $\phi_T(\nu[T, 1]) \cap \{|z| < R\}$ containing πi . Assume T is so large that $\phi_T(\nu[T, 1])$ intersects $\{|z| = R\}$ (this is possible by Lemma 5.4). Denote $e(T, R)$ the endpoint of $\tilde{\nu}$ and let T' be the corresponding time parameter, $\phi_T(\nu(T')) = e(T, R)$. If T is large enough, then the line segment $S = S(T, R) = [e(T, R), \phi_T(\hat{\nu}(T'))]$ joining e to the nearest point of the ‘‘previous turn’’ separates infinity from $\phi_T(\nu[T', 1])$ in $\phi_T(\mathbb{H} \setminus \nu[0, T'])$. By the monotonicity of the half plane capacity, we obtain

$$\text{hcap } k^{-1} \circ \psi_T(\tilde{\nu}) < \text{hcap } [L_T^{-1} \circ g_T(\nu(T, 1))/\sqrt{1-T}] < \text{hcap } k^{-1} \circ \psi_T(\tilde{\nu} \cup S).$$

By Lemma 5.6 and the continuity of capacity (Lemma 4.10) we see (by letting $R \rightarrow \infty$ as $T \rightarrow 1$) that

$$\text{hcap } k^{-1} \circ \psi_T(\tilde{\nu}) \rightarrow 1$$

as $T \rightarrow 1$. By the subadditivity of hcap ([La], Proposition 3.42) and $\text{hcap } k^{-1} \circ \psi_T(S) \rightarrow 0$, it follows that

$$\frac{1}{\alpha_T^2} = \text{hcap } [L_T^{-1} \circ g_T(\nu(T, 1))/\sqrt{1-T}] \rightarrow 1.$$

Fix $t < 1$. Then there is $R = R(t)$ (independent of T) such that $\phi_T(\nu[T, T + t(1 - T)]) \subset \tilde{\nu}(T, R)$: Indeed, denote $R(T, t)$ the largest R such that $\tilde{\nu}(T, R) \subset \phi_T(\nu[T, T + t(1 - T)])$, and assume to the contrary that there is no upper bound on $R(T, t)$ as $T \rightarrow 1$. Then the argument of the previous paragraph shows that

$$\limsup_{T \rightarrow 1} \text{hcap } k^{-1} \circ \psi_T(\tilde{\nu}(T, R(T, t))) = 1.$$

But

$$\text{hcap } k^{-1} \circ \psi_T(\tilde{\nu}(T, R(T, t))) = \text{hcap} [L_T^{-1} \circ g_T(\nu(T, T + t(1 - T)))/\sqrt{1 - T}] = \frac{t}{\alpha_T^2}$$

is bounded away from 1 as $T \rightarrow 1$, proving the existence of $R = R(t)$. (A direct estimate gives that $R(t)$ is comparable to $s = \log 1/(1 - t)$).

Thus Lemma 5.6 shows that ψ_T is uniformly close to the identity on $\phi_T(\nu[T, T + t(1 - T)])$, and it follows that $\nu_T(\tau) - \nu_T(0) + 4$ is uniformly close to $\gamma^4(\tau)$, on $\tau \in [0, t]$. Now uniform convergence of the driving term of $\nu_T(\tau) - \nu_T(0) + 4$ to the driving term of γ^4 is an easy consequence of Theorem 4.3. Indeed, using the notation of Theorem 4.3, fix $\tau \leq t$ and let g_1 and g_2 be the hydrodynamically normalized conformal maps associated with the curves $\gamma^4[0, \tau]$ and $\nu_T([0, \tau]) - \nu_T(0) + 4$. So g_1 is equal to g_τ from Section 3.3, Tangential intersection. Let $p_1 = \gamma^4(\tau)$, $p_2 = \nu_T(\tau) - \nu_T(0) + 4$, $p = \gamma^4((1 - \varepsilon)\tau + \varepsilon)$ and let σ_j be the line segment from p_j to p . The hyperbolic distance from ∞ to p is bounded independent of τ since $\tau \leq t < 1$. \square

Definition 5.8. We will say that a driving term $\mu : [0, 1) \rightarrow \mathbb{R}$ has local Lip 1/2 norm $\leq C$ if there is $\delta > 0$ such that

$$|\mu(t) - \mu(t')| \leq C|t - t'|^{1/2} \quad \text{for all } 0 \leq t < t' < 1 \quad \text{with } |t - t'| < \delta(1 - t). \quad (5.5)$$

We say that μ has arbitrarily small local Lip 1/2 norm, if for every $\epsilon > 0$, μ has local Lip 1/2 norm $\leq \epsilon$.

Proposition 5.9. If $\nu = \nu^A$ is the spiral constructed in Section 5.2, then its driving term $\lambda = \lambda^A$ has arbitrarily small local Lip 1/2 norm.

Proof. We need to show that the driving term of $g_t(\nu[t, t + \delta(1 - t)])$ has small Lip 1/2 norm. Since scaling does not change the Lip 1/2 norm, this is equivalent to saying that the renormalizations ν_t , restricted to the interval $[0, \delta] \subset [0, 1]$, have small Lip 1/2 norm if δ is small. Using the analyticity of the basic spiral ν_0 together with Koebe distortion, it is not hard to see that $\nu_t[0, \delta]$ is a $K(\delta)$ -quasislit half-plane with $K(\delta) \rightarrow 1$ as $\delta \rightarrow 0$. Now the proposition follows from Theorem 2 in [MR2]. \square

We end this section by noticing that the proofs of this section can be modified to show the following:

Theorem 5.10. If a sufficiently smooth (for instance asymptotically conformal) Loewner trace $\gamma[0, 1]$ has a self-intersection of angle $\pi(1 - \theta)$ (see Figure 2) with $\theta \in [0, 1)$, then

$$\lim_{t \rightarrow 1} \frac{\lambda(t) - \lambda(1)}{\sqrt{1 - t}} = \kappa,$$

where

$$\kappa = 2\sqrt{1 - \theta} + \frac{2}{\sqrt{1 - \theta}} > 4. \quad (5.6)$$

Similarly if γ is asymptotically similar to the logarithmic spiral S_θ (3.6) of Section 3.2 then

$$\lim_{t \rightarrow 1} \frac{\lambda(t) - \lambda(1)}{\sqrt{1 - t}} = \kappa,$$

where $\kappa = -4 \sin \theta$.

6 Collisions

In this section we give sufficient conditions for the trace to intersect itself in finite time.

Theorem 6.1. *Suppose $\lambda(t)$ is continuous on $[0, 1]$, satisfies*

$$\lim_{t \rightarrow 1} \frac{\lambda(t)}{\sqrt{1-t}} = \kappa > 4, \quad (6.1)$$

and assume there is $C < 4$ so that λ has local Lip 1/2 norm less than C (Definition 5.8). Then the trace $\gamma[0, 1]$ driven by λ is a Jordan arc. Moreover, $\gamma_T(1) \in \mathbb{R}$ and

$$\lim_{t \rightarrow 1} \arg(\gamma_T(t) - \gamma_T(1)) = \pi \frac{1 - \sqrt{1 - 16/\kappa^2}}{1 + \sqrt{1 - 16/\kappa^2}}, \quad (6.2)$$

provided $1 - T$ is sufficiently small.

Condition 5.5 with $C < 4$ is the smoothness condition referred to in Theorem 1.3. It will be used to prove that the trace γ_T , for T near 1, is a curve which is close to the self-similar curve given in Proposition 3.2. Recall that $\gamma_T = g_T(\gamma(T+t))/\sqrt{1-T}$, $t \in [0, 1-T]$. The reason that γ_T appears in the conclusion of Theorem 6.1 instead of γ is that the trace γ might intersect itself in \mathbb{H} rather than in \mathbb{R} . Alternatively, we could have added the requirement that $\|\lambda(t) - \kappa\sqrt{1-t}\|_\infty$ be sufficiently small and then the conclusion holds with γ_T replaced by γ , as in the statement of Theorem 1.3.

The method of proof also applies to the case $|\kappa| < 4$ and yields the following result:

Theorem 6.2. *Suppose $\lambda(t)$ is continuous on $[0, 1]$, satisfies*

$$\left| \lim_{t \rightarrow 1} \frac{\lambda(t)}{\sqrt{1-t}} \right| < 4, \quad (6.3)$$

and assume there is $C < 4$ so that λ has local Lip 1/2 norm less than C . Then the trace γ driven by λ is a Jordan arc. Moreover, γ is asymptotically similar to the logarithmic spiral at $\gamma(1) \in \mathbb{H}$.

We first outline the idea underlying the proofs of Theorems 6.1 and 6.2, then give the details of the proof of Theorem 6.1, and finally describe the adjustments necessary for the proof of Theorem 6.2.

Outline of the Proof of Theorems 6.1 and 6.2. Since λ has local Lip 1/2 norm less than 4, the trace $\gamma[0, t]$ is a Jordan arc for each $t < 1$ by Theorem 4.1. Let $\Gamma(s) = \gamma(t(s))$ be the reparametrization of γ described in Section 2.2. Let Γ^κ denote the self-similar curve driven by $\sigma^\kappa(s) \equiv \kappa$ as in Proposition 3.2, and let F^κ be the solution to (2.12) driven by σ^κ . Fix u_0 large and decompose Γ as

$$\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$$

where $\Gamma_n = \Gamma_n(\sigma) = \Gamma[(n-1)u_0, nu_0]$. Then

$$G_{(n-2)u_0}(\Gamma_n(\sigma)) = \Gamma_2(\sigma_{(n-2)u_0})$$

$(\sigma_{(n-2)u_0})$ is σ shifted by $(n-2)u_0$. By assumption, $\sigma_{(n-2)u_0}$ is close to κ if n is large, hence $G_{(n-2)u_0}(\Gamma_n(\sigma))$ is close to $\Gamma_2(\sigma^\kappa)$. Notice that $\Gamma_2(\sigma^\kappa)$ is close to a line segment if u_0 is large. Now

$$\Gamma_n = F_{(n-2)u_0}(G_{(n-2)u_0}(\Gamma_n(\sigma)))$$

so the Theorems follow from the fact that the map $F_{(n-2)u_0}$ is conformal (and contracting) in a neighborhood of the fixpoint B resp. β of F^κ , where the neighborhood does not depend on n . In the case $\kappa > 4$, this is proved in Lemma 6.3. If $\kappa < 4$, this follows because $\beta \in \mathbb{H}$ and all F_s are univalent in \mathbb{H} .

Now for the details.

Proof of Theorem 6.1. As before, let Γ^κ denote the self-similar curve driven by $\sigma^\kappa(s) \equiv \kappa$ as in Proposition 3.2, and let F^κ be the solution to (2.12) driven by σ^κ . Set $\theta = 2(1 + \kappa/\sqrt{\kappa^2 - 16})^{-1}$,

$$A = \frac{2}{\sqrt{1-\theta}} \quad \text{and} \quad B = 2\sqrt{1-\theta}, \quad (6.4)$$

so that $B < A < \kappa = A + B$. Then by Proposition 3.2, Γ^κ is a curve in \mathbb{H} from κ to B , which meets \mathbb{R} at angle $\pi(1-\theta) = \pi \frac{1-\sqrt{1-16/\kappa^2}}{1+\sqrt{1-16/\kappa^2}}$.

Let $\sigma(s) = e^{s/2}\lambda(1 - e^{-s})$ be the (time changed) driving term associated with λ and let Γ be the trace driven by σ . Our first task is to prove that the solutions F_s to the (time changed) Loewner equation (2.12) extend to be analytic in a fixed neighborhood of B .

Define the interval $I_s^\kappa = [x_1^\kappa(s), x_2^\kappa(s)] = G_s^\kappa(\Gamma^\kappa[0, s])$ as the preimage of $\Gamma^\kappa[0, s]$, by the map F_s^κ so that $F_s^\kappa(x_1^\kappa(s)) = F_s^\kappa(x_2^\kappa(s)) = \kappa$ and $F_s^\kappa(\kappa) = \Gamma(s)$. By the Schwarz Reflection Principle, F_s^κ extends to be a conformal map of $\mathbb{C} \setminus I_s^\kappa$ onto $\mathbb{C} \setminus (\Gamma^\kappa[0, s] \cup \Gamma^\kappa[0, s]^\mathbb{R})$ where $\Gamma^\kappa[0, s]^\mathbb{R}$ is the reflection of $\Gamma^\kappa[0, s]$ about \mathbb{R} . Note that by (3.2) and (3.3) we have that $F_s^\kappa(A) = A$. Since $F_s^\kappa(x_j) = \kappa$ and $F_s^\kappa(\kappa) = \Gamma^\kappa(s)$ is the tip of the slit $\Gamma^\kappa[0, s]$ we conclude

$$0 < B < A < x_1^\kappa < \kappa < x_2^\kappa < \infty.$$

See Figure 11.

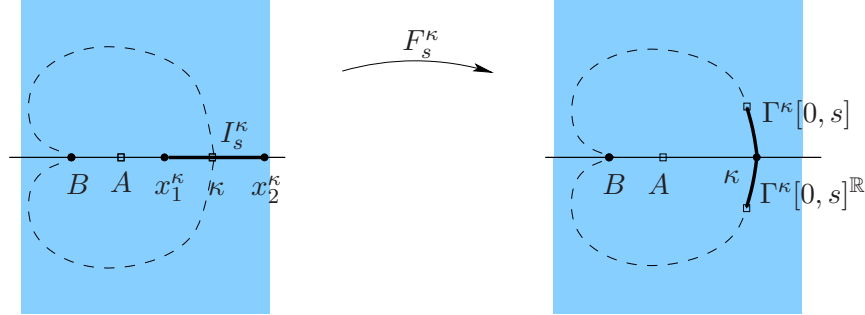


Figure 11: Extending the map F_s^κ by reflection.

Let $G_s = G_s^\sigma$ be the solution to (time changed) Loewner's differential equation (2.11) driven by σ and let $I_s = I_s^\sigma = [x_1(s), x_2(s)] = G_s(\Gamma[0, s])$.

Lemma 6.3. *Suppose $\kappa > 4$. Given $\delta > 0$ there exists $\epsilon_1 > 0$ so that if $\|\sigma - \kappa\|_\infty < \epsilon_1$ then*

$$I_s \subset (A - \delta, \infty) \quad (6.5)$$

where A is defined by (6.4).

Proof. Write $I_s = [x_1(s), x_2(s)]$. By (2.11)

$$\dot{G} = \frac{2}{G-\sigma} + \frac{G}{2} = \frac{G^2 - \sigma G + 4}{2(G-\sigma)}.$$

Thus $\dot{G} > 0$ whenever

$$\frac{\sigma - \sqrt{\sigma^2 - 16}}{2} < G < \frac{\sigma + \sqrt{\sigma^2 - 16}}{2} < \sigma. \quad (6.6)$$

Recall from (3.4) and (3.5) that $A + B = \kappa$ and $AB = 4$, so that A and B are roots of the equation $\zeta^2 - \kappa\zeta + 4 = 0$. Since $B < A < \kappa$, we may suppose that δ is so small that $B + \delta < A - \delta$. Then for ϵ_1 sufficiently small and $\|\sigma - \kappa\|_\infty < \epsilon_1$, we have that

$$\frac{\sigma - \sqrt{\sigma^2 - 16}}{2} < B + \delta < A - \delta < \frac{\sigma + \sqrt{\sigma^2 - 16}}{2}. \quad (6.7)$$

Thus $G_s(\kappa - \epsilon_1)$ is a continuous function of s with $G_s(\kappa - \epsilon_1) < x_1(s)$ and $G_0(\kappa - \epsilon_1) = \kappa - \epsilon_1 > A - \delta$. Suppose there is an $s > 0$ so that

$$G_s(\kappa - \epsilon_1) < A - \delta.$$

Then we can find an $s_1 > 0$ and $s_2 > s_1$ so that

$$G_s(\kappa - \epsilon_1) \geq A - \delta \quad (6.8)$$

for $0 \leq s \leq s_1$ and

$$B + \delta < G_s(\kappa - \epsilon_1) < A - \delta \quad (6.9)$$

for $s_1 < s \leq s_2$. But by (6.6) and (6.7), $\dot{G} > 0$ for $s_1 < s < s_2$. This contradicts (6.8) and (6.9) and so $A - \delta < G_s(\kappa - \epsilon_1) < x_1(s)$. This completes the proof of Lemma (6.3). \square

Remark 6.4. *There is no uniform upper bound on x_2 . The expansion of G_s about ∞ is given by*

$$G_s(z) = e^{s/2}z + \frac{2se^{s/2}}{z} + O\left(\frac{1}{z^2}\right).$$

Thus

$$x_2 - x_1 = |I_s| = 4C(I_s) = 4e^{s/2}C(\Gamma[0, s] \cup \Gamma[0, s]^\mathbb{R}),$$

where $C(E)$ denotes the logarithmic capacity of E and $\Gamma[0, s]^\mathbb{R}$ is the reflection of $\Gamma[0, s]$ about \mathbb{R} . Thus the length of I_s is finite, but it tends to ∞ as $s \rightarrow \infty$.

In particular each F_s is analytic on the ball $\{z : |z - B| < \frac{A-B}{2}\}$ for ϵ_1 sufficiently small.

To simplify the notation somewhat, we define

$$\Gamma_{u,v} = G_u(\Gamma[u, v]) \quad \text{and} \quad \Gamma_{u,v}(s) = G_u(\Gamma(u+s)), \quad (6.10)$$

for $0 \leq s \leq v - u$. Then $\Gamma_{u,v}(0) = G_u(\Gamma(u)) = \sigma(u)$ and $\Gamma_{u,v}(v - u) = G_u(\Gamma(v))$.

Lemma 6.5. *If (6.1) holds and if there is $C < 4$ so that λ has local Lip 1/2 norm less than C , then given $\varepsilon > 0$ and $0 < u_0 < \infty$, there is an $n_0 < \infty$ so that for $n \geq n_0$*

$$\rho_{\mathbb{H}}(\Gamma_{(n-1)u_0, (n+1)u_0}(s), \Gamma^\kappa(s)) < \varepsilon \quad (6.11)$$

for all $u_0 \leq s \leq 2u_0$, where $\rho_{\mathbb{H}}$ is the hyperbolic distance in the upper half-plane \mathbb{H} .

Proof. By (2.8) and (6.1), λ_T converges to $\lambda^\kappa(t) = \kappa\sqrt{1-t}$ uniformly on $[0, 1]$. Since the local Lip $1/2$ norm is less than $C < 4$, λ_T satisfies the hypotheses of Theorem 4.1 on $[0, t_0]$ for each $t_0 < 1$. By Theorem 4.1 this implies uniform convergence of $\gamma_T[0, t_0]$ to $\gamma^\kappa[0, t_0]$ for each $t_0 < 1$, as $T \rightarrow 1$. Since u_0 is fixed and Γ is a reparametrization of γ , the lemma follows. \square

Lemma 6.6. *For $u > 0$, let S_n be the line segment from $\Gamma(nu)$ to $\Gamma((n+1)u)$. Given $\epsilon > 0$, there is $n_0 < \infty$ and $u < \infty$ so that for $n \geq n_0$*

$$|\arg(\Gamma(nu) - \Gamma((n+1)u)) - \pi(1 - \theta)| < \epsilon, \quad (6.12)$$

$$\operatorname{Im}\Gamma((n+1)u) \leq \frac{1}{2}\operatorname{Im}\Gamma(nu), \quad (6.13)$$

and

$$\rho_{\mathbb{H}}(\Gamma(s), S_n) \leq \epsilon \quad (6.14)$$

whenever $nu \leq s \leq (n+1)u$, where $\rho_{\mathbb{H}}$ is the hyperbolic distance in \mathbb{H} .

Assuming Lemma 6.6 for the moment, we continue with the proof of Theorem 6.1. Set

$$C_\epsilon = \{z \in \mathbb{H} : |\arg z - \pi(1 - a)| < \epsilon\}$$

and

$$I_n = \mathbb{R} \cap [\Gamma(nu) - C_\epsilon].$$

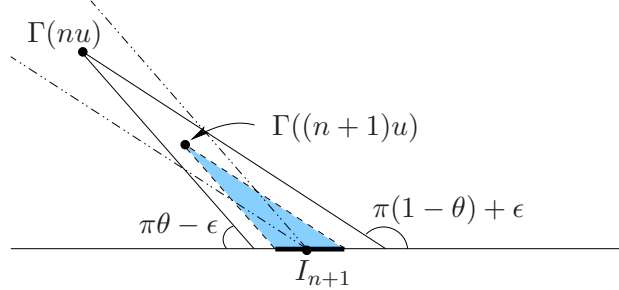


Figure 12: Cones with parallel sides

By (6.12), $\Gamma((n+1)u) \in \Gamma(nu) - C_\epsilon$ and since the cones $\Gamma(nu) - C_\epsilon$ and $\Gamma((n+1)u) - C_\epsilon$ have parallel sides, we conclude $I_{n+1} \subset I_n$. See Figure 12. By (6.13), $\operatorname{Im}\Gamma(nu) \rightarrow 0$ and hence $|I_n| \rightarrow 0$. Set

$$x_\infty = \bigcap I_n.$$

Note $x \in I_n$ if and only if $\Gamma(nu) \in x + C_\epsilon$. Thus $\Gamma(nu) \in x_\infty + C_\epsilon$ for all $n \geq n_0$. By (6.14),

$$\{\Gamma(s) : s > n_0u\} \subset x_\infty + C_{M\epsilon},$$

where M is a universal constant. Letting $\epsilon \rightarrow \infty$, we obtain the Theorem. \square

Proof of Lemma 6.6. To prove the lemma, we first verify that it holds for Γ^κ . As before F_s^κ is the inverse of $G_s = G_s^\kappa$. By (3.2) and (3.3)

$$k \circ F_s^\kappa = e^{\theta s/2} k(z)$$

and hence

$$\frac{(F_s^\kappa(z) - A)^{1-\theta}}{F_s^\kappa(z) - B} = e^{\theta s/2} \frac{(z - A)^{1-\theta}}{z - B}.$$

Since $F_s^\kappa(\kappa) = \Gamma^\kappa(s)$ we have that

$$\Gamma^\kappa(s) - B = e^{-\theta s/2} (\kappa - B) \left(\frac{\Gamma^\kappa(s) - A}{\kappa - A} \right)^{1-\theta} \quad (6.15)$$

Since $\kappa = A + B$, and $\Gamma^\kappa(s) \rightarrow B$ we have

$$\lim_{n \rightarrow \infty} \frac{\Gamma^\kappa(nu) - \Gamma^\kappa((n+1)u)}{e^{-nu\theta/2}} = A(1 - e^{-u\theta/2})(1 - A/B)^{1-\theta}.$$

Since $A > B$, (6.12) holds for n sufficiently large. Also (6.14) follows from (6.15). Choose u so large that $e^{-\theta u/2} < \frac{1}{2}$ and then (6.13) follows from (6.15) with $s = nu$. We also note that by (6.15)

$$|\arg(\Gamma^\kappa(s) - B) - \pi(1 - \theta)| < \epsilon \quad (6.16)$$

for $s \geq u$ if u is sufficiently large.

To prove the lemma for Γ , given $\epsilon > 0$, by Lemma 6.5 we can choose n_1 so large that if $n \geq n_1$ and $u \leq s \leq 2u$ then

$$\rho_{\mathbb{H}}(\Gamma_{(n-1)u, (n+1)u}(s), \Gamma^\kappa(s)) < \epsilon^2 \quad (6.17)$$

and by (6.16)

$$|\arg(\Gamma_{(n-1)u, (n+1)u}(s) - B) - \pi(1 - \theta)| < 2\epsilon. \quad (6.18)$$

Since $\Gamma^\kappa(s) \rightarrow B$ as $s \rightarrow \infty$ by (6.15), we can also choose u so large that

$$|\Gamma_{(n-1)u, (n+1)u}(s) - B| < \epsilon, \quad (6.19)$$

for $u \leq s \leq 2u$ and $n \geq n_1$, by Lemma 6.5 again.

Recall that by definition

$$\Gamma((n-1)u + s) = F_{(n-1)u}(\Gamma_{(n-1)u, (n+1)u}(s)) \quad (6.20)$$

for $0 \leq s \leq 2u$. Set

$$h = F_{(n-1)u}(z + B) - F_{(n-1)u}(B).$$

Then $h_1(z) = h(\frac{A-B}{2}z)$ is univalent on the unit disk \mathbb{D} by Lemma 6.3 and $h(0) = 0$. By (6.19), (6.18), (6.20), and Theorem 3.5, [D2, page 95], applied to $h_1/h_1'(0)$, we conclude that

$$|\arg(\Gamma(s) - F_{(n-1)u}(B)) - \pi(1 - \theta)| < 3\epsilon \quad (6.21)$$

for $nu \leq s \leq (n+1)u$. By (6.15) and (6.17),

$$\left| \frac{\Gamma_{(n-1)u, (n+1)u}(2u) - B}{\Gamma_{(n-1)u, (n+1)u}(u) - B} \right| < \frac{1}{2}. \quad (6.22)$$

By the upper and lower estimates in the growth theorem [GM, Theorem I.4.5],

$$\left| \frac{\Gamma((n-1)u + s_1) - F_{(n-1)u}(B)}{\Gamma((n-1)u + s_2) - F_{(n-1)u}(B)} \right| \leq (1 + \epsilon) \left| \frac{\Gamma_{(n-1)u, (n+1)u}(s_1) - B}{\Gamma_{(n-1)u, (n+1)u}(s_2) - B} \right|, \quad (6.23)$$

for $u \leq s_1, s_2 \leq 2u$. By (6.21), (6.22), and (6.23) with $s_1 = 2u$ and $s_2 = u$ we obtain (6.13) and then (6.12) for Γ . By (6.13), (6.21), and (6.23) we have that (6.14) holds for Γ . \square

Proof of Theorem 6.2. As before, fix u_0 large and write

$$\gamma[0, 1] = \bigcup_{n=1}^{\infty} \Gamma_n(s)$$

where $\Gamma_n = \Gamma_n(\sigma) = \Gamma[(n-1)u_0, nu_0]$. Since γ is continuous on $[0, 1)$ by the assumption $C < 4$ and [Li], we only need to show that $\text{diam } \gamma[t, 1] \rightarrow 0$ as $t \rightarrow 1$ in order to prove continuity of γ on $[0, 1]$. To do this, it suffices to show that $\text{diam } \Gamma_n$ decays exponentially. Notice that

$$\Gamma_n = F_{(n-2)u_0}(G_{(n-2)u_0}(\Gamma_n(\sigma))),$$

and write $F_{(n-2)u_0}$ as a composition

$$F_{(n-2)u_0} = f_1 \circ f_2 \circ \cdots \circ f_{n-2},$$

where each f_j corresponds to the driving term σ restricted to $[(j-1)u_0, ju_0]$. By choosing u_0 large enough, we may assume that all f_j except perhaps f_1 are arbitrarily close to $F_{u_0}^\kappa$ (driven by the constant $\sigma^\kappa \equiv \kappa$).

Writing $G = G^\kappa$, (3.8) implies

$$G'_s(\beta) = \left| \frac{1}{r} \right| = |e^{s(\cos \theta)e^{i\theta}}| > 1$$

so that

$$|F_{u_0}^{\kappa'}(\beta)| < 1.$$

Choosing u_0 large, Hurwitz' theorem implies that all f_j ($j \geq 2$) have a fixpoint β_j near the fixpoint β of $F_{u_0}^\kappa$, and we may assume that the derivatives $f'_j(\beta_j)$ are arbitrarily close to $F_{u_0}^{\kappa'}(\beta)$, hence uniformly bounded away from 1 in absolute value. As all f_j are conformal maps of \mathbb{H} , the Koebe distortion theorem (or normality) implies the existence of a disc D centered at β and a constant $c < 1$ such that $f_j(D) \subset D$ and $|f'_j(z)| \leq c$ for all $z \in D$ and all $j \geq 2$. Consequently,

$$\text{diam } F_{(n-2)u_0}(D) \leq c^{n-2}.$$

Since for u_0 large enough,

$$G_{(n-2)u_0}(\Gamma_n(\sigma)) = \Gamma_2(\sigma_{(n-2)u_0}) \subset D$$

for all n , it follows that

$$\Gamma_n = F_{(n-2)u_0}(\Gamma_2(\sigma_{(n-2)u_0})) \subset F_{(n-2)u_0}(D)$$

and the exponential decay of $\text{diam } \Gamma_n$ follows at once. By Theorem 4.1, $\Gamma_2(\sigma_{(n-2)u_0})$ converges to $\Gamma_2(\sigma^\kappa)$ as $n \rightarrow \infty$. Because $\Gamma(\sigma^\kappa)$ near β is asymptotically similar to the logarithmic spiral by Section 3.2 and Koebe distortion (applied to k^{-1} near 0), $\Gamma_2(\sigma^\kappa)$ rescaled (by a linear map) to have diameter 1 converges to (a portion of) the logarithmic spiral as $u_0 \rightarrow \infty$. Again by Koebe distortion (applied to $F_{(n-2)u_0}$) it follows that Γ_n rescaled to have diameter 1 converges to the spiral, and the theorem follows. \square

Proof of Theorem 1.2. Let γ be any of the spirals constructed in Section 5.2 and let λ be its driving term. By Theorem 5.3 and Proposition 5.9, $r\lambda$ satisfies the assumptions of Theorems 6.1 and 6.2 for $r > 1$ and $r < 1$ respectively, and Theorem 1.2 follows at once. \square

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