Angular Derivatives and Lipschitz Majorants

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ABSTRACT: It is an old problem, dating back at least to Ahlfors [1930], to give geometric conditions on the boundary of a simply connected domain Ω near $\zeta \in \partial\Omega$ so that a conformal map ψ of Ω onto the unit disk or half plane extends to be "conformal" at ζ, in the sense that ψ has a non-zero angular derivative at ζ. In this paper, we solve a problem of Burdzy [1986] and [1987, page 164] by giving geometric conditions for a certain class of regions. The main results, Theorem 9(ii) and Theorem 13(ii) are complementary to Burdzy's work in [1986], and extend earlier work of Rodin and Warschawski [1977]. We also give a classical analysis proof of Burdzy's Theorem in Theorem 13(i). The history of this subject is extensive. The interested reader might begin with Warschawski [1967], Rodin-Warschawski [1977] , Baernstein [1988], and the references therein. We will first review material needed to understand our result.

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§1 Background.

Throughout this paper, Ω will denote a simply connected domain in the complex plane and a map defined on Ω will be called *conformal* if it is one-to-one and analytic.

Definition 1. *We say that $\partial\Omega$ has an* **inner tangent** *at $\zeta \in \partial\Omega$ if there is an angle θ_0 so that for every $\beta \in (0, \pi/2)$ there is an $\varepsilon = \varepsilon(\beta) > 0$ and so that the truncated cone*

$$\Gamma_\beta^\varepsilon(\zeta) = \{z : |\arg(z-\zeta) - \theta_0| < \beta,\ 0 < |z-\zeta| < \varepsilon\}$$

is contained in Ω. When $\theta_0 = \pi/2$ we say that Γ has **a vertical inner normal**.

Definition 2. *If $\partial\Omega$ has an inner tangent at $\zeta \in \partial\Omega$ and if φ is a conformal map defined on Ω, then we say φ is* **semi-conformal** *at ζ if φ has a non-tangential limit*

$$\varphi(\zeta) \equiv \lim_{\Gamma_\beta^\varepsilon(\zeta) \ni z \to \zeta} \varphi(z)$$

for every $\beta \in (0, \pi/2)$ and if

$$A_\zeta = \lim_{\Gamma_\beta^\varepsilon(\zeta) \ni z \to \zeta} \arg \frac{\varphi(z) - \varphi(\zeta)}{z - \zeta} \tag{1}$$

exists for every $\beta \in (0, \pi/2)$.

If φ is semi-conformal at ζ then for $|\alpha - \theta_0| < \pi/2$, the image of the ray $\{z : \arg(z-\zeta) = \alpha\}$ is asymptotic to the ray $\{z : \arg(w - \varphi(\zeta)) = A_\zeta + \alpha\}$, as $z \to \zeta$. Thus $\partial\varphi(\Omega)$ has an inner tangent at $\varphi(\zeta)$ and φ^{-1} is semi-conformal at $\varphi(\zeta)$. If φ is a conformal map of the unit disk $\mathbb{D}$ onto a region Ω bounded by a Jordan curve Γ and if Γ has a tangent at $w \in \Gamma$, then by Carathéodory's theorem and Lindelöf's theorem (see e.g. Pommerenke [1975]), φ is semi-conformal at $\zeta = \varphi^{-1}(w)$; moreover convergence $z \to \zeta$ is not restricted to cones. However in general, φ can be semi-conformal at $\zeta \in \partial\mathbb{D}$ even though Γ does not have a tangent at $\varphi(\zeta)$.

The next theorem, due to Ostrowski, gives geometric conditions on region $\Omega = \varphi(\mathbb{H})$ equivalent to the semi-conformality of φ at ζ, where $\mathbb{H}$ denotes the upper half plane $\{z : \operatorname{Im} z > 0\}$. For convenience, we will state the case when $\zeta = 0$, $\varphi(0) = 0$ and when the limit $A_\zeta = 0$ in (1); for the other cases, simply translate and rotate.

Theorem 3. (Ostrowski [1937]). *Suppose Ω is a simply connected domain in $\mathbb{C}$. If the conformal map φ of the upper half-plane $\mathbb{H}$ onto Ω is semi-conformal at 0 with $\varphi(0) = 0$, and if the limit in (1) is 0, then $\partial\Omega$ has an inner tangent at 0 with a vertical inner normal and*

$$\lim_{\mathbb{R} \ni x \to 0} \frac{\operatorname{dist}(x, \partial\Omega)}{x} = 0. \tag{2}$$

Conversely, if $\partial\Omega$ has an inner tangent at 0 *with a vertical inner normal and satisfies (2) then we can choose the conformal map φ of $\mathbb{H}$ onto Ω so that it is semi-conformal at* 0 *with non-tangential limit equal to* 0 *and so that the limit in (1) is 0.*

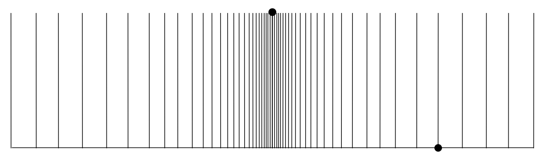

Figure 1

The "tines" of the comb in Figure 1 are located at $x_n - i$ with $x_n = -x_{-n} \to 0$, $n \in \mathbb{Z}$. If φ is a conformal map of the upper half plane onto the complement of the comb (including ∞) with $\varphi(0) = 0$, then by Ostrowski's theorem φ is semi-conformal at 0 if and only if $\lim_{n\to\pm\infty} x_{n+1}/x_n = 1$.

Definition 4. *If $\partial\Omega$ has an inner tangent at $\zeta \in \partial\Omega$ and if φ is a conformal map defined on Ω, then we say φ has* **angular derivative** *$\varphi'(\zeta)$ at ζ if φ has a non-tangential limit*

$$\varphi(\zeta) \equiv \lim_{\Gamma^e_\beta(\zeta)\ni z\to\zeta} \varphi(z)$$

for every $\beta \in (0, \pi/2)$ and if

$$\varphi'(\zeta) \equiv \lim_{\Gamma^e_\beta(\zeta)\ni z\to\zeta} \frac{\varphi(z) - \varphi(\zeta)}{z - \zeta}$$

exists for every $\beta \in (0, \pi/2)$.

It is not hard to show that φ has an angular derivative $\varphi'(\zeta)$ at ζ if and only if φ' has non-tangential limit $\varphi'(\zeta)$. See for example, Pommerenke [1975].

If φ has a *non-zero* angular derivative at ζ then φ is semi-conformal at ζ, and φ^{-1} has a non-zero angular derivative at $\varphi(\zeta)$. Preserving angles in a *region* Ω is equivalent to having a non-zero derivative in Ω, however it is possible for φ to be semi-conformal at $\zeta \in \partial\Omega$ and not have an angular derivative at ζ. A normal families argument does, however, give us some information about the modulus of the difference quotient if φ is semi-conformal at ζ. For if $\partial\Omega$ has an inner tangent with vertical inner normal at $0 \in \partial\Omega$ and if φ is conformal on Ω and semi-conformal at 0, consider the functions

$$f_r(z) = \log \frac{\varphi(rz) - \varphi(0)}{rz} \cdot \frac{ri}{\varphi(ri) - \varphi(0)}$$

Then $\mathrm{Im} f_r$ converges to 0, as $r \to 0$, uniformly on compact subsets of the half-annulus

$$A = \{z : 1/2 < |z| < 2\} \cap \mathbb{H}.$$

Since $\mathrm{Re} f_r(i) = 0$,

$$\lim_{r\to 0} \left| \frac{\varphi(rz) - \varphi(0)}{rz} \right| \cdot \left| \frac{ri}{\varphi(ri) - \varphi(0)} \right| = \lim_{r \to 0} e^{\mathrm{Re} f_r} = 1 \tag{3}$$

uniformly on compact subsets of A. Thus if φ is semi-conformal at ζ, then φ has an angular derivative at 0 if and only if

$$\lim_{r\to 0} \left| \frac{\varphi(ri) - \varphi(0)}{ri} \right| \tag{4}$$

exists. This says that that there may be stretching or compression, but only in the radial direction.

The **Angular Derivative Problem** is to give Euclidean geometric conditions on the boundary of a simply connected region Ω near $\zeta \in \partial\Omega$ which are equivalent to the existence of a non-zero angular derivative at ζ for the conformal map of Ω onto the half-plane or the disc. The existence of a non-zero angular derivative only depends on the geometry of Ω near ζ and does not depend on the choice of the conformal map.

Definition 5. *If Ω is simply connected and if $0 \in \partial\Omega$, we say Ω has a* **positive angular derivative** *at 0 if there is a conformal map φ of $\mathbb{H}$ onto Ω which has non-tangential limit $\varphi(0) = 0$ and which has angular derivative $\varphi'(0)$ with $0 < \varphi'(0) < \infty$.*

By the remarks above, a characterization of simply connected domains with positive angular derivative at 0 would solve the Angular Derivative Problem.

The angular derivative problem has been converted into a problem about extremal length, which can be estimated in many cases by the geometry. If E and F are subsets of the closure, $\overline{\Omega}$, of Ω, let Γ denote the collection of locally rectifiable curves in Ω connecting E to F. The **extremal distance** in Ω between E and F is defined to be

$$d_\Omega(E,F) = \sup_{\rho \geq 0} \frac{\left(\inf_{\gamma\in\Gamma} \int_\gamma \rho |dz|\right)^2}{\int_\Omega \rho^2 dA}, \tag{5}$$

where $|dz|$ denotes arc length measure, dA denotes Lebesgue area measure and where the supremum is taken over all non-negative Borel functions ρ satisfying $0 < \int_\Omega \rho^2 dA < \infty$. (Such functions ρ are called **metrics**.) For example, the extremal distance between the vertical ends of a rectangle, with sides parallel to the axes, is the ratio of the length to the height. Since extremal distance is conformally invariant, if $E_r = \{z : |z| = r\} \cap \mathbb{H}$, then for $t > s$, $d_\Omega(E_t, E_s) = \frac{1}{\pi} \log \frac{t}{s}$, where Ω is the half-annulus $\{z : s < |z| < t,\ \mathrm{Im} z > 0\}$. The region Ω can also be taken to be the upper half plane $\mathbb{H}$. See Ahlfors [1973] for an introduction to extremal length.

Theorem 6. (Jenkins-Oikawa [1977], Rodin-Warschawski [1976]) *Suppose* Ω *is a simply connected domain containing the positive imaginary axis, with* $0 \in \partial\Omega$. *Let* E_r *denote the component of* $\{z : |z| = r\} \cap \Omega$ *containing* ir, *for* $r > 0$. *Then* Ω *has a positive angular derivative at* 0 *if and only if*

(i) Ω has an inner tangent at 0 with a vertical inner normal and

(ii) $\lim_{s<t\to 0} d_\Omega(E_t, E_s) - \frac{1}{\pi}\log\frac{t}{s} = 0.$

Condition (ii) says that the conformal map behaves like a constant multiple of z, asymptotically. Note also that if (2) holds then by the argument used to establish (4), we can add the further restriction to condition (ii) that $s = 2^{-n}$ and $t = 2^{-m}$, $n, m \in \{1, 2, 3, \ldots\}$ and the theorem still holds.

§2 Strip Domains.

It is convenient sometimes to transform both Ω and $\mathbb{H}$ to "strips". The "standard strip"

$$\mathbb{S} = \{z : |\mathrm{Im} z| < \pi/2\}$$

is mapped onto the upper half-plane $\mathbb{H}$ by the function $\tau(z) = ie^{-z}$. Then

$$\psi(z) = \tau^{-1} \circ \varphi \circ \tau(z) = \pi i/2 - \log\varphi(ie^{-z}) \tag{6}$$

is a conformal map of $\mathbb{S}$ onto a region $\widetilde{\Omega}$ with

$$\Omega = \{ie^{-z} : z \in \widetilde{\Omega}\}.$$

Cones are replaced by half-strips

$$S_\delta = \{z : |\mathrm{Im} z| < \pi/2 - \delta \text{ and } \mathrm{Re} z > 0\},$$

for $\delta \in (0, \pi/2)$. Thus $\partial\Omega$ has an inner tangent at 0 with a vertical normal if and only if for every $\delta \in (0, \pi/2)$ there is and $x_\delta > 0$ so that

$$x_\delta + S_\delta \subset \widetilde{\Omega}. \tag{7}$$

The conformal map φ of $\mathbb{H}$ onto Ω has a non-tangential limit 0 at $\zeta = 0$ if and only if

$$\mathrm{Re}\psi(z) \to +\infty \text{ as } \mathrm{Re} z \to +\infty,\ z \in S_\delta \tag{8}$$

for every $\delta \in (0, \pi/2)$. In this case, φ is semi-conformal at 0 if and only if

$$\lim_{\substack{\mathrm{Re}z \to +\infty \\ z \in S_\delta}} \mathrm{Im}\psi(z) - \mathrm{Im}z \tag{9}$$

exits for every $\delta \in (0, \pi/2)$. Condition (2) in Ostrowski's theorem is equivalent to

$$\lim_{\substack{\mathrm{Re}z \to +\infty \\ z \in \partial \mathbb{S}}} \mathrm{dist}(z, \partial\widetilde{\Omega}) = 0. \tag{10}$$

Likewise φ has non-tangential limit 0 and non-zero angular derivative at 0 if and only if

$$\lim_{\substack{\mathrm{Re}z \to +\infty \\ z \in S_\delta}} \psi(z) - z \tag{11}$$

exists for every $\delta \in (0, \pi/2)$.

It has become the custom to consider slightly more general circumstances, by removing the restriction that τ be one-to-one.

Definition 7. *If $\widetilde{\Omega}$ is simply connected and satisfies (7), let ψ be a conformal map of $\widetilde{\Omega}$ onto $\mathbb{S}$ which satisfies (8). The region $\widetilde{\Omega}$ is said to have an* **angular derivative (at $+\infty$)** *when (11) holds.*

Clearly this definition does not depend on the choice of the map ψ and hence is a property of the region $\widetilde{\Omega}$. A geometric characterization of regions with angular derivative at $+\infty$ would give a solution to the Angular Derivative Problem. Theorem 3 extends to this slightly more general context, if we use (9) as the definition of semi-conformality and replace (2) with (10). Likewise Theorem 6 extends by replacing (i) with (7) and replacing (ii) with

$$\lim_{s,t \to +\infty} d_{\widetilde{\Omega}}(F_s, F_t) - |t - s|/\pi = 0, \tag{12}$$

where F_x denotes the component of $\{z \in \widetilde{\Omega} : \mathrm{Re}z = x\}$ containing x. This more general form of Theorem 6, using (7) and (12), is in Jenkins-Oikawa [1977] and Rodin-Warschawski [1976].

For strip domains, there are some classical estimates of extremal distance which motivate our results. If E and F are the vertical sides of the rectangle

$$R = \{z : |\mathrm{Im}z| < H,\ |\mathrm{Re}z| < L\},$$

then the extremal distance satisfies

$$d_R(E, F) = L/H.$$

By the conformal invariance of extremal distance, if ψ is a conformal map of a region U onto R with $\psi(E') = E$ and $\psi(F') = F$, then

$$d_U(E', F') = L/H$$

and the metric $\rho = |\psi'| = |\nabla \mathrm{Re}\psi|$ is extremal in the sense that the supremum in (5) is achieved with this metric.

Any metric ρ gives a lower bound for extremal distance. To motivate a choice of a metric below, we first consider the special case of a quadrilateral

$$U = \{(x, y) : |y - m(x)| < \theta(x)/2,\ a < x < b\},$$

of width $\theta(x)$ and mid-line $y = m(x)$. Let $\sigma_j = U \cap \{z : \mathrm{Re} z = x_j\}$, $j = 0, \ldots, n$, where $a = x_0 < x_1 < \ldots < x_n = b$. By the serial rule (see Ahlfors [1973]),

$$d_U(\sigma_0, \sigma_n) \geq \sum_{j=1}^{n} d(\sigma_{j-1}, \sigma_j).$$

The region between σ_{j-1} and σ_j is approximately at thin rectangle, if $\Delta x = x_j - x_{j-1}$ is small. This rectangle can be mapped to a rectangle of height 1, centered on $\mathbb{R}$ by the linear map

$$\frac{z - im(x_j)}{\theta(x_j)}.$$

Since the image rectangle has width $\Delta x/\theta(x_j)$, this suggests the (non-analytic) map

$$(x, y) \xrightarrow{\Phi} \left(\int_a^x \frac{dt}{\theta(t)}, \frac{y - m(x)}{\theta(x)} \right) \tag{13}$$

of U onto a rectangle of height 1 and length $\int_a^b \frac{1}{\theta(t)} dt$. The extremal metric for $d_U(\sigma_0, \sigma_n)$ is given by $\rho(z) = |\varphi'(z)| = |\nabla \mathrm{Re}\varphi(z)|$ where φ is a conformal map of U onto a rectangle, so let

$$\rho(z) = |\nabla \mathrm{Re}\Phi| = \frac{1}{\theta(x)}$$

where $z = x + iy$. If γ is any curve connecting σ_0 to σ_n in U, then

$$\int_\gamma \rho(z)|dz| \geq \int_a^b \frac{1}{\theta(x)} dx.$$

Moreover,

$$\iint_U \rho^2 dy dx = \int_a^b \frac{1}{\theta(x)} dx$$

and so

$$d_U(\sigma_0, \sigma_n) \geq \int_a^b \frac{1}{\theta(x)} dx. \tag{14}$$

More generally, suppose $\widetilde{\Omega}$ is a simply connected domain containing the real line $\mathbb{R}$. Let F_x denote the component of $\{z \in \widetilde{\Omega} : \text{Re}z = x\}$ containing x. Fix $s < t < \infty$, and define a metric ρ_A by

$$\rho_A(x, y) = \begin{cases} \frac{1}{\theta(x)} & \text{if } s < x < t \text{ and } (x, y) \in F_x \\ 0 & \text{elsewhere in } \widetilde{\Omega}. \end{cases}$$

Then by the argument above

$$d_{\widetilde{\Omega}}(F_s, F_t) \geq \int_s^t \frac{1}{\theta(x)} dx. \tag{15}$$

This estimate is due to Ahlfors.

For regions of the form $U = \{(x, y) : |y - m(x)| < \theta(x)/2,\ a < x < b\}$ there is an upper bound for $d_U(\sigma_0, \sigma_n)$ that complements (14). Notice that the extremal distance between the two *horizontal* sides of the rectangle R, given above, is equal to the reciprocal of the extremal distance between the vertical sides, by a rotation. Any metric gives a lower bound for the extremal distance between the two components of $\partial U \setminus (\sigma_0 \cup \sigma_n)$ and thus gives an upper bound for the extremal distance between σ_0 and σ_n.

By (13), a natural choice for a metric is then

$$\rho_B(x, y) = \left| \nabla \left(\frac{y - m(x)}{\theta(x)} \right) \right| = \sqrt{\frac{1}{\theta^2(x)} + \left(\frac{(y - m)\theta' + \theta m'}{\theta^2} \right)^2}. \tag{16}$$

This metric was discovered by Beurling [1989].

If γ is any curve in U connecting the curve $y = m(x) + \theta(x)/2$ to the curve $y = m(x) - \theta(x)/2$, then

$$\int_\gamma \rho_B |dz| \geq \left| \int_\gamma \nabla \left(\frac{y - m}{\theta} \right) \cdot dz \right| = 1. \tag{17}$$

Furthermore,

$$\int_U \rho_B^2 dA = \int_a^b \int_{m - \frac{\theta}{2}}^{m + \frac{\theta}{2}} \left\{ \frac{1}{\theta^2} + \left[\frac{(y - m)\theta' + \theta m'}{\theta^2} \right]^2 \right\} dy dx$$

$$= \int_a^b \frac{dx}{\theta(x)} + \int_a^b \frac{m'(x)^2 + \frac{1}{12}\theta'(x)^2}{\theta(x)} dx.$$

Thus

$$d_U(\sigma_0, \sigma_n) \le \int_a^b \frac{dx}{\theta(x)} + \int_a^b \frac{m'(x)^2 + \frac{1}{12}\theta'(x)^2}{\theta(x)} dx.$$

For regions of the form $\widetilde{\Omega} = \{x + iy : |y - m(x)| < \theta(x)/2\}$, let $F_x = \{z \in \widetilde{\Omega} : \mathrm{Re} z = x\}$. Then for $s < t$,

$$d_{\widetilde{\Omega}}(F_s, F_t) \le \int_s^t \frac{dx}{\theta(x)} + \int_s^t \frac{m'(x)^2 + \frac{1}{12}\theta'(x)^2}{\theta(x)} dx. \tag{18}$$

The inequality (18) was discovered by Warschawski [1942], using the older length-area method.

These estimates of Ahlfors and Beurling can be used to give necessary and sufficient geometric conditions for the existence of a positive angular derivative when $\partial\Omega$ is sufficiently smooth. See Rodin-Warschawski [1976].

Corollary 8. *Suppose $\widetilde{\Omega}$ is a simply connected domain given by*

$$\widetilde{\Omega} = \{x + iy : |y - m(x)| < \theta(x)/2\}$$

where $\lim_{x \to +\infty} m(x) = 0$ and

$$\int_0^\infty (m')^2 + (\theta')^2 dx < \infty. \tag{19}$$

Then $\widetilde{\Omega}$ has an angular derivative at $+\infty$ if and only if

$$\int_0^\infty \left(\frac{1}{\theta(x)} - \frac{1}{\pi} \right) dx \tag{20}$$

exists.

Proof. If (20) holds then $\lim_{s \to +\infty} \theta(s) = \pi$. Indeed, if $\varepsilon > 0$, then for s and t sufficiently large, with $s < t < s + 1$, by the Cauchy-Schwarz inequality and (19)

$$|\theta(s) - \theta(t)| < \varepsilon.$$

If there is a sequence $s_n \to +\infty$ with $\theta(s_n) > \pi + 2\varepsilon$, then $\theta(t) > \pi + \varepsilon$ for $s_n < t < s_n + 1$ and hence

$$\int_{s_n}^{s_n+1} \left(\frac{1}{\theta(t)} - \frac{1}{\pi} \right) dt < -\frac{2\varepsilon}{\pi(\pi + 2\varepsilon)}$$

contradicting (20). A similar argument works if there is a sequence $s_n \to +\infty$ with $\theta(s_n) < \pi - 2\varepsilon$. Thus (7) and (10) hold. As before, let $F_x = \widetilde{\Omega} \cap \{z : \mathrm{Re} z = x\}$. For s, t sufficiently large, by (15) and (18) we have the estimates

$$\int_s^t \frac{1}{\theta(x)} dx \le d_{\widetilde{\Omega}}(F_r, F_s) \le \int_s^t \frac{1}{\theta(x)} dx + \int_s^t \frac{m'(x)^2 + \frac{1}{12}\theta'(x)^2}{\theta(x)} dx.$$

Thus by the strip version of Theorem 6, $\widetilde{\Omega}$ has a angular derivative at $+\infty$ if and only if

$$\lim_{s,t\to+\infty}\int_s^t\left(\frac{1}{\theta(x)}-\frac{1}{\pi}\right)dx=0. \tag{21}$$

□

§3 Area Estimates.

Condition (21) has a more geometric interpretation. If $\widetilde{\Omega}$ is a region containing the real line $\mathbb{R}$, let F_r denote the component of $\{z:\mathrm{Re}z=r\}\cap\widetilde{\Omega}$ containing $r\in\mathbb{R}$, and let

$$\widetilde{\Omega}'=\text{interior}\left(\bigcup_{r\in\mathbb{R}}F_r\right).$$

For any region U, let

$$U_{s,t}=U\cap\{z:s<\mathrm{Re}z<t\}.$$

Then the integral in (21) is related to the area measure of the difference between the standard strip $\mathbb{S}$ and the region $\widetilde{\Omega}'$ since

$$\begin{aligned}\int_s^t\left(\frac{1}{\theta}-\frac{1}{\pi}\right)dx&\geq\frac{1}{\pi^2}\int_s^t(\pi-\theta)dx\\&=\frac{1}{\pi^2}\left(\mathrm{Area}(\mathbb{S}\setminus\widetilde{\Omega}')_{s,t}-\mathrm{Area}(\widetilde{\Omega}'\setminus\mathbb{S})_{s,t}\right).\end{aligned}\tag{22}$$

Thus if $\widetilde{\Omega}$ is a region containing $\mathbb{R}$ then by (15)

$$d_{\widetilde{\Omega}}(F_s,F_t)-(t-s)/\pi\geq\frac{1}{\pi^2}\left(\mathrm{Area}(\mathbb{S}\setminus\widetilde{\Omega}')_{s,t}-\mathrm{Area}(\widetilde{\Omega}'\setminus\mathbb{S})_{s,t}\right). \tag{23}$$

If $\theta(x)\to\pi$ as $x\to+\infty$ (as was the case in Corollary 8) then the inequality (22) is almost an equality as $s,t\to\infty$. For example if

$$\mathrm{Area}(\mathbb{S}\setminus\widetilde{\Omega}')+\mathrm{Area}(\widetilde{\Omega}'\setminus\mathbb{S})<\infty$$

then

$$\lim_{s,t\to\infty}\int_s^t\left|\frac{1}{\theta(x)}-\frac{1}{\pi}\right|dx=0.$$

The above discussion becomes even more significant if we apply it not to $\widetilde{\Omega}$ but to a smaller region with Lipschitz boundary. If $\widetilde{\Omega}$ is a region containing $\mathbb{R}$, let $\mathcal{T}_M$ denote the collection of isosceles triangles contained in $\widetilde{\Omega}$ with base on $\mathbb{R}$ and sides with slope $\pm M$. Set

$$\widetilde{\Omega}_M=\bigcup\{T:T\in\mathcal{T}_M\}.$$

An M-Lipschitz curve is the graph of a function h which satisfies

$$|h(s) - h(t)| \leq M|s - t|$$

for all $s, t \in \mathbb{R}$. Then Ω_M is the largest subregion of $\widetilde{\Omega}$ containing $\mathbb{R}$ which is bounded by M-Lipschitz curves.

Theorem 9. *Suppose $\widetilde{\Omega}$ is a simply connected domain with $\mathbb{R} \subset \widetilde{\Omega}$ and*

$$\partial\widetilde{\Omega} \cap \{z : \mathrm{Re} z < 0\} = \partial\mathbb{S} \cap \{z : \mathrm{Re} z < 0\}. \tag{24}$$

Let $\widetilde{\Omega}_M$ be the largest region bounded by M-Lipschitz curves contained in $\widetilde{\Omega}$, as defined above.

(i) If $\mathrm{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M) < \infty$ then $\widetilde{\Omega}$ has an angular derivative at $+\infty$ if and only if $\mathrm{Area}(\widetilde{\Omega}_M \setminus \mathbb{S}) < \infty$.

(ii) Suppose $M > 8\pi$. If $\mathrm{Area}(\widetilde{\Omega}_M \setminus \mathbb{S}) < \infty$, then $\widetilde{\Omega}$ has an angular derivative at $+\infty$ if and only if $\mathrm{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M) < \infty$.

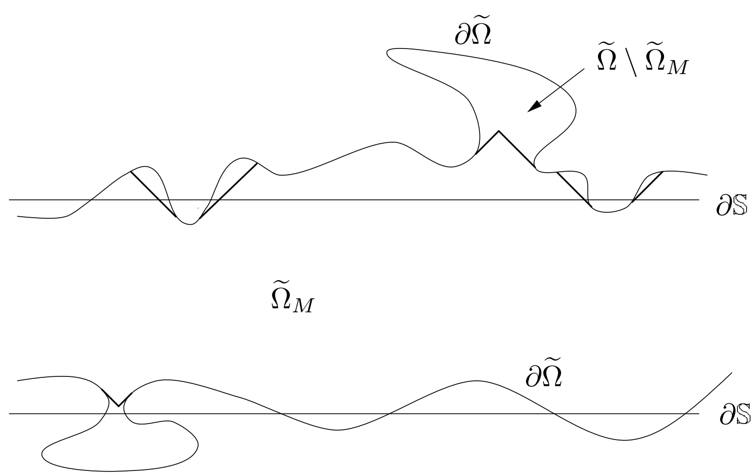

Figure 2

The statement (24) is inconsequential since the existence of an angular derivative at $+\infty$ is a local property of $\partial\Omega$ near $+\infty$.

Statement (i) was proved by Burdzy [1986] in the form given in Theorem 13(i) below using Brownian excursions. See also Carroll [1988] and Gardiner [1991]. Half of statement (i), in the form given above, was also proved by Rodin-Warschawski [1986] and the other half was also proved by Sastry [1995]. Statement (ii) and the proof of (i) below are new. Since there seems to be some uncertainty (see e.g. Sastry [1995]) about the exact relationship between Theorem 9(i) and Theorem 13(i), we will first prove Theorem 9, then derive Theorem 13 from Theorem 9. If $\tau(z) = ie^{-z}$ is one-to-one on $\widetilde{\Omega}$ (see section 2) then one can derive Theorem 9 (for $\widetilde{\Omega}$) from Theorem 13 in a similar manner, so long as 8π is replaced by a somewhat larger constant.

Proof. If $\text{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M) < \infty$ and if $\text{Area}(\widetilde{\Omega}_M \setminus \mathbb{S}) < \infty$ then

$$\lim_{\substack{\text{Re}z \to +\infty \\ z \in \partial\widetilde{\Omega}_M}} \text{dist}(z, \partial\mathbb{S}) = 0, \tag{25}$$

and hence (7) holds. Conversely, if $\widetilde{\Omega}$ has an angular derivative at $+\infty$, then by Theorem 3, (25) holds. Thus we may suppose for the remainder of the proof that (25) holds. If F_s denotes the component of $\widetilde{\Omega} \cap \{z : \text{Re}z = s\}$ containing $s \in \mathbb{R}$, then by (25) and the strip version of Theorem 6, $\widetilde{\Omega}$ has an angular derivative at $+\infty$ if and only if

$$\lim_{\substack{s,t \to \infty \\ s < t}} d_{\widetilde{\Omega}}(F_s, F_t) - (t-s)/\pi = 0. \tag{26}$$

To prove Theorem 9, we will obtain lower and upper bounds for the left side of (26) in Lemma 10 and Lemma 12 below.

Suppose that $\partial\widetilde{\Omega}_M$ is given by the two curves $y = h_1(x) + \pi/2$ and $y = -h_2(x) - \pi/2$, $-\infty < x < +\infty$, where $h_1 > -\pi/2$ and $h_2 > -\pi/2$. By (25), $\lim_{x \to +\infty} h_j(x) = 0$. As before, for any region U, let

$$U_{s,t} = U \cap \{z : s < \text{Re}z < t\}.$$

Lemma 10. (Sastry [1995]) *Suppose* $|h_j(x)| \le \pi M^2$ *for* $s < x < t$. *Then*

$$d_{\widetilde{\Omega}}(F_s, F_t) - (t-s)/\pi \le \tfrac{4M^2+2}{\pi^2} \text{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M)_{s,t} - \tfrac{1}{(4M^2+2)\pi^2} \text{Area}(\widetilde{\Omega}_M \setminus \mathbb{S})_{s,t}.$$

Sastry [1995] proved this lemma with different constants and used a piecewise constant metric. We will use a metric of the form $\rho = |\nabla u|$, since it is much easier to estimate lengths as we have seen for example in (17).

Proof. Since $d_{\widetilde{\Omega}}(F_s, F_t) \le d_{\widetilde{\Omega}_M}(F_s, F_t) = d_{(\widetilde{\Omega}_M)_{s,t}}(F_s, F_t)$, without loss of generality, we may suppose (for notational convenience) that $\widetilde{\Omega} = (\widetilde{\Omega}_M)_{s,t}$. Set $\lambda = 1/(2M^2)$ and define

$$U_1 = \{(x,y) \in \widetilde{\Omega} : h_1(x) > 0 \text{ and } -\lambda h_1 + \pi/2 < y < h_1 + \pi/2\}$$

$$U_2 = \{(x,y) \in \widetilde{\Omega} : h_1(x) < 0 \text{ and } 2h_1 + \pi/2 < y < h_1 + \pi/2\}$$

$$U_3 = \{(x,y) \in \widetilde{\Omega} : h_2(x) > 0 \text{ and } -h_2 - \pi/2 < y < \lambda h_2 - \pi/2\}$$

$$U_4 = \{(x,y) \in \widetilde{\Omega} : h_2(x) < 0 \text{ and } -h_2 - \pi/2 < y < -2h_2 - \pi/2\} \text{ and}$$

$$U^* = \widetilde{\Omega} \setminus \cup_{j=1}^4 U_j.$$

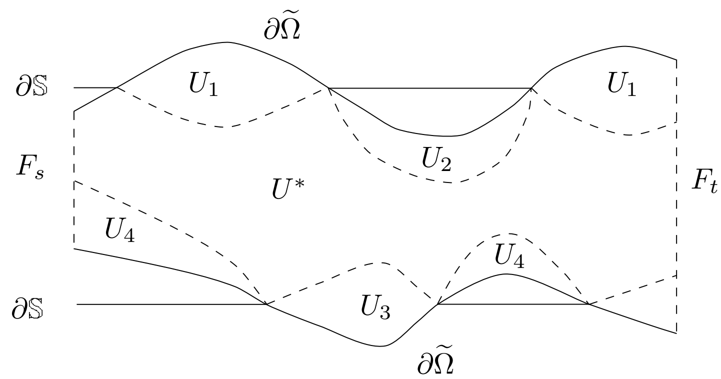

Figure 3

Consider the continuous map $(x,y)\to(x,u(x,y))$ of $\widetilde{\Omega}$ onto $\mathbb{S}$ which fixes U^* and is linear in y on $\widetilde{\Omega}\setminus U^*$. Thus

$$u(x,y)=\begin{cases} y & \text{for } (x,y)\in U^* \\ (y-(h_1+\pi/2))/(2M^2+1)+\pi/2 & \text{for } (x,y)\in U_1 \\ 2(y-(h_1+\pi/2))+\pi/2 & \text{for } (x,y)\in U_2 \\ (y+h_2+\pi/2)/(2M^2+1)-\pi/2 & \text{for } (x,y)\in U_3 \\ 2(y+h_2+\pi/2)-\pi/2 & \text{for } (x,y)\in U_4. \end{cases}$$

Set $\rho=|\nabla u|$ on $\widetilde{\Omega}$. This metric is similar to Beurling's metric in section 2, except that we fix the region U^*. Note that

$$|\nabla u|^2\le\begin{cases} 1 & \text{on } U^* \\ 4(1+M^2) & \text{on } U_2\cup U_4\ . \\ (1+M^2)/(2M^2+1)^2 & \text{on } U_1\cup U_3 \end{cases}$$

Thus

$$\begin{aligned}\int_{\widetilde{\Omega}}|\nabla u|^2dydx-\int_{\mathbb{S}_{s,t}}dydx &\le 4(1+M^2)\mathrm{Area}(U_2\cup U_4)+\frac{1+M^2}{(2M^2+1)^2}\mathrm{Area}(U_1\cup U_3)\\ &\quad-\frac{1}{2M^2}\mathrm{Area}(\widetilde{\Omega}\setminus\mathbb{S}_{s,t})-2\mathrm{Area}(\mathbb{S}_{s,t}\setminus\widetilde{\Omega})\\ &=(4M^2+2)\mathrm{Area}(\mathbb{S}_{s,t}\setminus\widetilde{\Omega})-\frac{1}{4M^2+2}\mathrm{Area}(\widetilde{\Omega}\setminus\mathbb{S}_{s,t}).\end{aligned}$$

If γ is a curve connecting $\{y=h_1(x)-\pi/2\}$ to $\{y=-h_2(x)-\pi/2\}$ and contained in $\widetilde{\Omega}$, then

$$\int_\gamma|\nabla u||dz|\ge\left|\int_\gamma\nabla u\cdot dz\right|=\pi.$$

Thus

$$d_{\widetilde{\Omega}}(F_s,F_t) \le \frac{\int_{\widetilde{\Omega}} |\nabla u|^2 dydx}{(\inf_\gamma \int_\gamma |\nabla u||dz|)^2}$$
$$\le \frac{1}{\pi^2}\left(\pi(t-s) + (4M^2+2)\text{Area}(\mathbb{S}_{s,t}\setminus\widetilde{\Omega}) - \frac{1}{4M^2+2}\text{Area}(\widetilde{\Omega}\setminus\mathbb{S}_{s,t})\right).$$

□

The next step is to derive a lower bound for extremal distance. Write $\partial\widetilde{\Omega}_M \setminus \partial\widetilde{\Omega} = \cup_j \sigma_j$ where each σ_j is an open arc with endpoints $z_j^l, z_j^r \in \partial\widetilde{\Omega}$ where $\text{Re}z_j^l < \text{Re}z_j^r$. For $v \in \mathbb{C}\setminus\mathbb{R}$, let T_v denote the triangle in $\mathcal{T}_M$ with vertex v. There is a unique v_j with $\text{Re}z_j^l \le \text{Re}v_j \le \text{Re}z_j^r$ so that $z_j^l, z_j^r \in \partial T_{v_j}$. Then $\sigma_j \subset \partial T_{v_j}$ and σ_j consists of at most two line segments. If $|\text{Im}z_j^l| \le |\text{Im}z_j^r|$, let $B_j = \{z : |z - z_j^l| < \text{Re}z_j^r - \text{Re}z_j^l\}$. Otherwise let $B_j = \{z : |z - z_j^r| < \text{Re}z_j^r - \text{Re}z_j^l\}$.

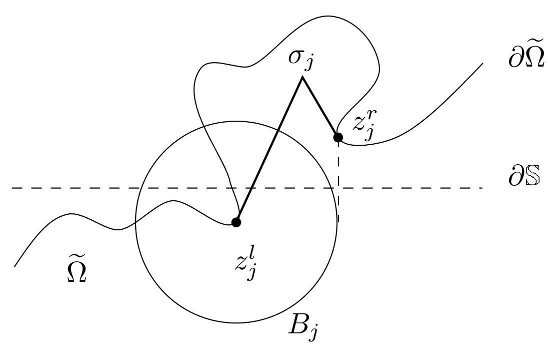

Figure 4

Lemma 11.

$$\text{Area}(B_j) \le \frac{8\pi}{M}\int_{z\in\sigma_j}\left||\text{Im}z| - \frac{\pi}{2}\right| dx, \qquad z = x+iy.$$

Proof. Suppose f is defined and continuous on $[0,1]$ with

$$f'(x) = \begin{cases} 1 & 0 \le x \le a \\ -1 & a \le x \le 1. \end{cases}$$

Then by elementary calculus

$$\int_0^1 |f(x)|dx \ge \frac{1}{8}.$$

If $\text{Im}z > 0$ on σ_j, the map

$$(x,y) \to \left(\frac{x - \text{Re}z_j^l}{\text{Re}z_j^r - \text{Re}z_j^l}, \frac{y-\pi/2}{M(\text{Re}z_j^r - \text{Re}z_j^l)}\right)$$

transforms σ_j into the graph of one such f. Thus

$$\int_{z\in\sigma_j} |\mathrm{Im}z - \frac{\pi}{2}| dx \geq \frac{M(\mathrm{Re}z_j^r - \mathrm{Re}z_j^l)^2}{8} = \frac{M}{8\pi}\mathrm{Area}(B_j).$$

The inequality of the lemma for $\mathrm{Im}z < 0$ follows by a reflection about $\mathbb{R}$. □

The quantity

$$\int_{z\in\sigma_j} \left||\mathrm{Im}z| - \frac{\pi}{2}\right| dx$$

is the total area "between" σ_j and $\partial\mathbb{S}$.

The next lemma gives a lower bound for the extremal distance in $\widetilde{\Omega}$ using the geometry of $\widetilde{\Omega}_M$. When $\widetilde{\Omega} = \widetilde{\Omega}_M$, it follows immediately from (23).

Lemma 12. *If $\varepsilon > 0$ there is an $s_0 < \infty$ so that for $s < t < \infty$,*

$$d_{\widetilde{\Omega}}(F_s, F_t) - (t-s)/\pi \geq \tfrac{M-8\pi}{M\pi^2}\mathrm{Area}(\mathbb{S}\setminus\widetilde{\Omega}_M)_{s,t} - \tfrac{M+8\pi}{M\pi^2}\mathrm{Area}(\widetilde{\Omega}_M\setminus\mathbb{S})_{s,t} - \varepsilon.$$

Proof. If $0 \leq t - s \leq \varepsilon$, then by (25) $\mathrm{Area}(\widetilde{\Omega}_M\setminus\mathbb{S})_{s,t} \to 0$ and $\mathrm{Area}(\mathbb{S}\setminus\widetilde{\Omega}_M)_{s,t} \to 0$ as $s, t \to \infty$, and the inequality follows. Now suppose that $t - s \geq \varepsilon$. Let ρ be the metric on $\widetilde{\Omega}$ given by

$$\rho = \begin{cases} 1 & \text{for } z \in \widetilde{\Omega}_M \cup \bigcup_j B_j \\ 0 & \text{elsewhere on } \widetilde{\Omega}. \end{cases}$$

The metric ρ will provide the lower bound for the extremal distance. First we compute the ρ-area of $(\widetilde{\Omega}_M)_{s,t}$. By Lemma 11

$$\begin{aligned}\int_{(\widetilde{\Omega}_M)_{s,t}} \rho^2 dxdy - \int_{\mathbb{S}_{s,t}} dxdy &\leq \mathrm{Area}(\widetilde{\Omega}_M\setminus\mathbb{S})_{s,t} - \mathrm{Area}(\mathbb{S}\setminus\widetilde{\Omega}_M)_{s,t} + \sum_j \mathrm{Area}(B_j) \\ &\leq (1+\frac{8\pi}{M})\mathrm{Area}(\widetilde{\Omega}_M\setminus\mathbb{S})_{s,t} - (1-\frac{8\pi}{M})\mathrm{Area}(\mathbb{S}\setminus\widetilde{\Omega}_M)_{s,t}.\end{aligned} \tag{27}$$

Now suppose γ is a curve in $\widetilde{\Omega}$ connecting F_s to F_t, with $s < t$. Note that

$$\mathrm{Re}z_j^r - \mathrm{Re}z_j^l \leq \frac{4}{M}\max_{z\in\sigma_j}||\mathrm{Im}z| - \pi/2| \to 0 \tag{28}$$

as $\sigma_j \to +\infty$, by (25). Thus by deleting an initial and terminal portion of γ, if necessary, we may suppose that γ begins and ends in $\widetilde{\Omega}_M \cup \bigcup_j B_j$ and that if γ meets σ_j, then $s < \mathrm{Re}z_j^l < \mathrm{Re}z_j^r < t$.

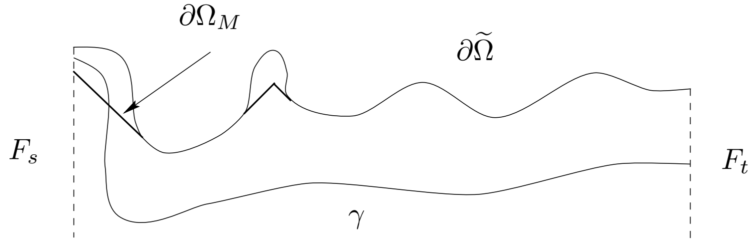

Figure 5

Thus we can write $\gamma = \cup_k \gamma_k$ where $\{\gamma_k\}$ are disjoint subarcs of γ such that either $\gamma_k \subset \widetilde{\Omega}_M \cup \bigcup_j B_j$, or γ_k meets some σ_j and connects $F_{\mathrm{Re} z_j^l}$ to $F_{\mathrm{Re} z_j^r}$ with $s < \mathrm{Re} z_j^l < \mathrm{Re} z_j^r < t$. If $\gamma_k \subset \widetilde{\Omega}_M \cup \bigcup_j B_j$, then $\int_{\gamma_k} \rho |dz|$ is at least equal to the change in Rez along γ_k. If γ_k connects $F_{\mathrm{Re} z_j^l}$ to $F_{\mathrm{Re} z_j^r}$ and intersects σ_j, then

$$\mathrm{length}(\gamma_k \cap (\widetilde{\Omega}_M \cup B_j)) \geq \mathrm{Re} z_j^r - \mathrm{Re} z_j^l.$$

Indeed, if z_j^l is the center of B_j and if $\zeta \in \gamma_k \cap \sigma_j$, then the shortest curve from $F_{\mathrm{Re} z_j^l}$ to ζ in $\widetilde{\Omega}$ is the straight line from z_j^l to ζ, since $|\mathrm{Im}\zeta| \geq |\mathrm{Im} z_j^l|$. Since γ_k must intersect ∂B_j, it must have length at least the radius of B_j, namely $\mathrm{Re} z_j^r - \mathrm{Re} z_j^l$. A similar argument works if z_j^r is the center of B_j.

Together with (28) this implies, for s, t sufficiently large, that

$$\int_\gamma \rho |dz| \geq t - s - \varepsilon.$$

We conclude by (27) that for $t - s \geq \varepsilon$,

$$d_{\widetilde{\Omega}}(F_s, F_t) \geq \frac{(t-s-\varepsilon)^2}{\pi(t-s) + (1+8\pi/M)\mathrm{Area}(\widetilde{\Omega}_M \setminus \mathbb{S})_{s,t} - (1-8\pi/M)\mathrm{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M)_{s,t}}.$$

Since $\partial\widetilde{\Omega}_M$ is Lipschitz and (25) holds, we have both $\mathrm{Area}(\widetilde{\Omega}_M \setminus \mathbb{S})_{s,t}/(t-s) \to 0$ and $\mathrm{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M)_{s,t}/(t-s) \to 0$ as $s, t \to \infty$. Hence for s, t sufficiently large with $t - s \geq \varepsilon$,

$$d_{\widetilde{\Omega}}(F_s, F_t) - (t-s)/\pi \geq \tfrac{M-8\pi}{M\pi^2}\mathrm{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M)_{s,t} - \tfrac{M+8\pi}{M\pi^2}\mathrm{Area}(\widetilde{\Omega}_M \setminus \mathbb{S})_{s,t} - \varepsilon.$$

□

We can now conclude the proof of Theorem 9. The condition $\mathrm{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M) < \infty$ is equivalent to $\mathrm{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M)_{s,t} \to 0$ as $s, t \to +\infty$. Likewise, the condition $\mathrm{Area}(\widetilde{\Omega}_M \setminus \mathbb{S}) < \infty$ is equivalent to

$\text{Area}(\widetilde{\Omega}_M \setminus \mathbb{S})_{s,t} \to 0$ as $s,t \to +\infty$. Thus if $\text{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M) < \infty$ and $\text{Area}(\widetilde{\Omega}_M \setminus \mathbb{S}) < \infty$, by Lemma 10 and Lemma 12,

$$\lim_{s,t\to\infty} d_{\widetilde{\Omega}}(F_s, F_t) - (t-s)/\pi = 0,$$

and hence by the discussion at the beginning of the proof of the Theorem 9, $\widetilde{\Omega}$ has an angular derivative at $+\infty$.

If $\text{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M) < \infty$ and if $\widetilde{\Omega}$ has an angular derivative at $+\infty$, then by (26) and Lemma 10, $\text{Area}(\widetilde{\Omega}_M \setminus \mathbb{S})_{s,t} \to 0$ as $s,t \to +\infty$, proving statement (i) of Theorem 9.

If $M > 8\pi$ and if $\text{Area}(\widetilde{\Omega}_M \setminus \mathbb{S}) < \infty$ then by (26) and Lemma 12, $\text{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M)_{s,t} \to 0$ as $s,t \to +\infty$, completing the proof of Theorem 9. $\square$

Theorem 9 does not solve the Angular Derivative Problem, even for regions bounded by Lipschitz curves. For example, if $\widetilde{\Omega} = \{(x,y) : |y - m(x)| < 1/2\}$ is a strip of constant width with $|m'(x)| \leq 1$ and

$$\int_0^\infty |m(x)| dx = \infty$$

then $\widetilde{\Omega}$ is bounded by two Lipschitz curves, but $\text{Area}(\mathbb{S} \setminus \widetilde{\Omega}) = \text{Area}(\widetilde{\Omega} \setminus \mathbb{S}) = \infty$. Thus Theorem 9 does not apply. However if we also have $m \to 0$ and

$$\int_0^\infty |m'(x)|^2 dx < \infty,$$

then by Corollary 8, $\widetilde{\Omega}$ has an angular derivative at $+\infty$.

Theorem 9 can be restated using Lipschitz majorants for the boundary of Ω with $0 \in \partial\Omega$. Suppose Ω is a plane domain. Let

$$\Gamma_M = \{z : \text{Im} z > M \text{Re} z\}$$

be the cone with vertex at 0 and with sides of slope $\pm M$, and let

$$\Omega^M = \bigcup \{\zeta + \Gamma_M : \zeta + \Gamma_M \subset \Omega\}.$$

If Ω^M is not empty, it is bounded by the graph of an M-Lipschitz function h_M. Set $h_M = +\infty$ if $\Omega^M = \emptyset$. We will call h_M the *smallest M-Lipschitz majorant of $\partial\Omega$*, for if the graph of an M-Lipschitz function h is contained in Ω then $h \geq h_M$.

Theorem 13. *Let Ω be a simply connected domain with $0 \in \partial\Omega$ and let h_M denote the smallest M-Lipschitz majorant of $\partial\Omega$.*

(i) (Burdzy [1986]). *Suppose*

$$\int_{-1}^{1} \chi_{h_M>0} \frac{h_M(x)}{x^2} dx < \infty. \tag{29}$$

Then Ω has a positive angular derivative at 0 if and only if

$$\int_{-1}^{1} \chi_{h_M<0} \frac{h_M(x)}{x^2} dx > -\infty. \tag{30}$$

(ii) There is an $M_0 < \infty$ so that if $M > M_0$ and if h_M satisfies (30) then Ω has a positive angular derivative at 0 if and only if h_M satisfies (29).

Theorem 13(ii) answers a question in Burdzy [1986] and Burdzy [1987] (Problem 11.7, page 164). In the next section we will give an example (suggested by Burdzy) where Theorem 9 (ii) and Theorem 13 (ii) fail for small M_0.

Proof. Since the existence of an angular derivative is a local property depending only on Ω near 0, we may suppose that $\partial\Omega \cap \{z : |z| > 1\} = \{x \in \mathbb{R} : |x| > 1\}$. If h_M satisfies (29) and (30) then $h_M(x)/x \to 0$ as $x \to 0$. For if $h_M(x_n) > \varepsilon|x_n| > 0$ then $h_M(x) > \varepsilon|x_n/2|$ on an interval centered at x_n of length $|x_n|\varepsilon/M$. This contradicts the integrability condition (29) if $x_n \to 0$. A similar argument holds if $h_M(x_n) < -\varepsilon|x_n| < 0$ using condition (30). Thus $\partial\Omega$ has an inner tangent at 0 with a vertical inner normal.

Conversely, if Ω has a positive angular derivative at 0 then by Theorem 3, $\partial\Omega$ has an inner tangent with a vertical inner normal and (2) holds. This implies $h_M(x)/x \to 0$ as $x \to 0$. Thus we may suppose, for the remainder of the proof that $h_M(x)/x \to 0$ as $x \to 0$.

Let γ_M denote the graph of h_M. If $N > M$, then $\gamma_M \cap \partial\Omega \subset \gamma_N \cap \partial\Omega$ and $h_N \leq h_M$. Write

$$\gamma_M \setminus \partial\Omega = \cup_j \sigma_j.$$

As in the proof of Theorem 9, each σ_j consists of two intervals, the leftmost with slope $-M$ and the rightmost with slope M, and endpoints $z_j^l, z_j^r \in \partial\Omega$. For $N > M$ there is a unique cone $\zeta_j + \Gamma_N$ with $z_j^l, z_j^r \in \zeta_j + \partial\Gamma_N$. Moreover the graph of $h_N(x)$ lies above $\zeta_j + \partial\Gamma_N$ and below σ_j, for $\mathrm{Re} z_j^l \leq x \leq \mathrm{Re} z_j^r$. Thus $\max|h_N| \leq C \max|h_M|$, for $\mathrm{Re} z_j^l \leq x \leq \mathrm{Re} z_j^r$, where C is a constant depending only on M and N. In particular, for $N > M$, $h_N(x)/x \to 0$ as $x \to 0$.

Suppose first that (29) holds and that Ω has a positive angular derivative at 0. Since

$$\lim_{x\to 0} h_M(x)/x = 0,$$

the graph of h_M, for small x, is transformed by the map

$$\varphi(z) = \pi i/2 - \log z$$

to two M' Lipschitz curves, where M' is a constant depending only on M. Set $\widetilde{\Omega} = \varphi(\Omega)$. These curves must then be contained in the region $\widetilde{\Omega}_{M'} \supset \widetilde{\Omega}_M$. Since $h_M(x)/x \to 0$, condition (29) implies

$$\lim_{s,t\to+\infty} \text{Area}(\mathbb{S}_{s,t} \setminus W) = 0,$$

where W is the image of the region $\{x+iy : |x| < 1, y > \max(h_m(x), 0)\}$ by the map φ. Since $W \subset \widetilde{\Omega}_{M'}$, (29) implies

$$\text{Area}(\mathbb{S} \setminus \widetilde{\Omega}_{M'}) < \infty.$$

By Theorem 9 (i), $\text{Area}(\widetilde{\Omega}_{M'} \setminus \mathbb{S}) < \infty$. Since $\varphi^{-1}(\widetilde{\Omega}_{M'})$ contains the graph of h_M, and since $h_M(x)/x \to 0$, as $x \to 0$, we must have (30).

There is a constant M_0 so that the region above the graph of h_M, $M \geq M_0$, contains $\varphi^{-1}(\widetilde{\Omega}_N)$ where $N = 8\pi + 1$. Suppose now that $M \geq M_0$, that condition (30) holds and that Ω has a positive angular derivative at 0. As above this implies $\text{Area}(\widetilde{\Omega}_N \setminus \mathbb{S}) < \infty$. By Theorem 9, $\text{Area}(\mathbb{S} \setminus \widetilde{\Omega}_N) < \infty$. As above, this implies (29).

Finally suppose that both (29) and (30) hold.

Lemma 14. *Suppose $0 < M < N < \infty$ and suppose f and g are continuous on $[a,b]$ with $f(a) = g(a)$, $f(b) = g(b)$,*

$$f'(x) = \begin{cases} -M & \text{for } a \leq x \leq c \\ M & \text{for } c \leq x \leq b \end{cases}$$

and

$$g'(x) = \begin{cases} -N & \text{for } a \leq x \leq d \\ N & \text{for } d \leq x \leq b \end{cases}$$

There is a constant C depending only on M and N so that if $g \leq h \leq f$ on $[a,b]$, then

$$\int_a^b |h(x)|dx \leq C \int_a^b |f(x)|dx.$$

Before proving the lemma, let us use it to complete the proof of Theorem 13. As above, let γ_M and γ_N denote the graphs of h_M and h_N, respectively, where $N > M$. Write $\gamma_M = (\gamma_M \cap \partial\Omega) \cup_j \sigma_j$

where each σ_j consists of two line segments of slope $\pm M$ on the interval $I_j = [a_j, b_j]$. As noted above, the endpoints of σ_j belong to $\partial\Omega$.

By the lemma with $a = a_j$, $b = b_j$, $f = h_M$ and $h = h_N$ we have

$$\int_{I_j} |h_N(x)|dx \leq C \int_{I_j} |h_M(x)|dx.$$

Since $h_M(x)/x \to 0$ as $x \to 0$, we must have $(b_j - a_j)/a_j \to 0$ as $b_j \to 0$, and hence

$$\int_{I_j} \frac{|h_N(x)|}{x^2}dx \leq \left(\frac{b}{a}\right)^2 \int_{I_j} \frac{|h_N(x)|}{b^2}dx \leq C\left(\frac{b}{a}\right)^2 \int_{I_j} \frac{|h_M(x)|}{x^2}dx.$$

Since $\gamma_M \cap \partial\Omega \subset \gamma_N \cap \partial\Omega$, we have $h_N = h_M$ on $\mathbb{R} \setminus \cup_j I_j$. Thus

$$\int_{-1}^{1} \frac{|h_N(x)|}{x^2}dx \leq C \int_{-1}^{1} \frac{|h_M(x)|}{x^2}dx.$$

In other words, (29) and (30) hold with M replaced by N, for all $N > M$.

Choose M' and $N > M'$ so that $\varphi^{-1}(\widetilde{\Omega}_{M'})$ contains the graph of h_M and lies above the graph of h_N. By (29) Area$(\mathbb{S} \setminus \widetilde{\Omega}_{M'}) < \infty$ and by (30) with M replaced by N, Area$(\widetilde{\Omega}_{M'} \setminus \mathbb{S}) < \infty$. By Theorem 9, Ω has a positive angular derivative at 0. □

Proof of Lemma 14. If $\max|f| \geq 2M(b-a)$ then $\min|f| \geq \max|f| - M(b-a) \geq \frac{1}{2}\max|f|$, and hence

$$\int_a^b |f(x)|dx \geq \frac{1}{2}\max|f|(b-a).$$

If $\max|f| < 2M(b-a)$ then as in the proof of Lemma 11,

$$\int_a^b |f(x)|dx \geq \frac{1}{8}M(b-a)^2 \geq \frac{1}{16}\max|f|(b-a).$$

Since $\max|f| \leq \max|g| \leq 4\frac{N}{M}\max|f|$ and $g \leq h \leq f$, we have

$$|h| \leq \frac{4N}{M}\max|f|.$$

We conclude that

$$\int_a^b |h|dx \leq \frac{4N}{M}\max|f|(b-a) \leq \frac{64N}{M}\int_a^b |f(x)|dx.$$

□

Corollary 15. *Let Ω be a simply connected domain containing the upper half plane $\mathbb{H}$, with $0 \in \partial\Omega$ and let h_M be the smallest M-Lipschitz majorant of $\partial\Omega$. Then Ω has a positive angular derivative at 0 if and only if*

$$\int_{-1}^{1} \frac{h_M(x)}{x^2}dx > -\infty.$$

This Corollary follows immediately from Theorem 13(i). See also Rodin-Warschawski [1977] for other equivalent conditions.

Corollary 16. *Let Ω be a simply connected domain contained in the upper half plane $\mathbb{H}$ with $0 \in \partial\Omega$. Suppose the smallest M-Lipschitz majorant of $\partial\Omega$, h_M, is not identically $+\infty$. Then Ω has a positive angular derivative at 0 if and only if*

$$\int_{-1}^{1} \frac{h_M(x)}{x^2} dx < \infty.$$

This Corollary follows immediately from Theorem 13(ii) and Proposition 17 below. See also Rodin-Warschawski [1977] for other equivalent conditions. Each of these Corollaries has a version for strip regions that follows immediately from Theorem 9.

§4 **Further Results.**

Burdzy (private communication) suggested the following example where Theorem 9 (ii) fails for small M. A similar example fails for the half plane version, Theorem 13. Suppose $0 < \varepsilon_n \to 0$, and $\sum \varepsilon_n^2 = \infty$. For $\delta > 0$, let

$$h_n(x) = \begin{cases} x & \text{for } 0 \le x \le \varepsilon_n \\ 2\varepsilon_n - x & \text{for } \varepsilon_n \le x \le 2\varepsilon_n + \delta \\ x - 2(\varepsilon_n + \delta) & \text{for } 2\varepsilon_n + \delta \le x \le 2(\varepsilon_n + \delta) \\ 0 & \text{elsewhere on } \mathbb{R}. \end{cases}$$

Then the curve $y = h_n(x)$ is 1-Lipschitz. Set

$$\widetilde{\Omega}_n = \{(x, y) : h_n(x) - \pi/2 < y < \pi/2\}$$

and set $s = -1$ and $t = +1$. Using the constant metric $\rho \equiv 1$, we obtain the lower bound

$$d_{\widetilde{\Omega}_n}(F_s, F_t) - (t - s)/\pi \ge \frac{(\varepsilon_n^2 - \delta^2)2}{\pi(2\pi - \varepsilon_n^2 + \delta^2)}.$$

By Lemma 10, since $\partial\Omega$ is 1-Lipschitz,

$$d_{\widetilde{\Omega}_n}(F_s, F_t) - (t - s)/\pi \le \frac{6}{\pi^2}\varepsilon_n^2 - \frac{1}{6\pi^2}\delta^2.$$

Thus we can choose $\delta = \delta_n$ with $\varepsilon_n \le \delta_n \le 6\varepsilon_n$ so that for $s = -1$ and $t = +1$,

$$d_{\widetilde{\Omega}_n}(F_s, F_t) = (t - s)/\pi.$$

We can then choose the conformal map φ_n of $\widetilde{\Omega}_n$ onto $\mathbb{S}$ so that φ_n converges to the identity map, uniformly on

$$\left(\overline{\mathbb{S}} \setminus \{|z+\pi i/2| < 1/K\}\right) \cap \{|z| < K\}$$

for all $K > 0$.

Define

$$h(x) = h_{n_j}(x - 2n_j)$$

for $2n_j - 1 < x < 2n_j + 1$, $j = 1, 2, \ldots$, set $h(x) = 0$ elsewhere in $\mathbb{R}$ and set

$$\widetilde{\Omega} = \{(x,y) : h(x) - \pi/2 < y < \pi/2\}.$$

Then we can choose $n_j \to \infty$ so that $\widetilde{\Omega}$ has an angular derivative at $+\infty$. Note that for $M \leq 1/13$,

$$\widetilde{\Omega}_M \subset \mathbb{S}$$

and

$$\text{Area}(\mathbb{S} \setminus \widetilde{\Omega}_M) = \sum \varepsilon_n^2 = +\infty.$$

Thus Theorem 9 (ii) cannot hold with $M = 1/13$.

Lemma 14 can be improved to give the next Proposition.

Proposition 17. *Let Ω be a simply connected domain with $0 \in \partial\Omega$ and for $M < \infty$, let h_M denote the smallest M-Lipschitz majorant of $\partial\Omega$. Then*

$$\int_{-1}^{1} \frac{|h_M(x)|}{x^2} dx < \infty$$

holds for some $M > 0$ if and only if it holds for all $M > 0$.

Proposition 17 follows from the next Lemma as in the proof of Theorem 9. A similar statement holds for strip regions using areas.

Lemma 18. *Suppose $0 < M < N < \infty$ and suppose f and g are continuous on $[a,b]$ with $f(a) = g(a)$, $f(b) = g(b)$,*

$$f'(x) = \begin{cases} -M & \text{for } a \leq x \leq c \\ M & \text{for } c \leq x \leq b \end{cases}$$

and

$$g'(x) = \begin{cases} -N & \text{for } a \leq x \leq d \\ N & \text{for } d \leq x \leq b \end{cases}$$

There is a constant C depending only on M and N so that if $g \le h \le f$ on $[a,b]$, then

$$\frac{1}{C}\int_a^b |f(x)|dx \le \int_a^b |h(x)|dx \le C\int_a^b |f(x)|dx.$$

Proof. The upper bound follows from Lemma 14. To prove the lower bound, first suppose that $\min g \le 0$. Write

$$\{x \in (a,b) : g(x) > 0\} = (a, a_1) \cup (a_4, b),$$

and

$$\{x \in (a,b) : f(x) < 0\} = (a_2, a_3),$$

where $a \le a_1 \le a_2 \le a_3 \le a_4 \le b$. Set

$$h_1(x) = \begin{cases} g(x) & \text{if } g(x) > 0 \\ 0 & \text{if } g(x) < 0 < f(x) \\ f(x) & \text{if } f(x) < 0. \end{cases}$$

Then

$$\int_a^b |h(x)|dx \ge \int_a^b |h_1(x)|dx.$$

Thus we may assume $h = h_1$, and the result reduces to a problem comparing areas of triangles. Note

$$\int_a^{a_2} |h_1(x)|dx = \int_a^{a_1} g(x)dx = \frac{|f(a)|^2}{2N} = \frac{M}{N}\int_a^{a_2} |f(x)|dx.$$

A similar inequality holds for the (possibly empty) interval (a_3, b). Since $|h_1| = |f|$ on (a_2, a_3), the lower bound follows in this case with $1/C = M/N$.

The second case to consider is when $\min g \ge 0$. In this case

$$\int_a^b |h(x)|dx \ge \int_a^b |g(x)|$$

and the latter integral is at least one half of the area of the trapezoid with base $[a,b]$ and sides of height $f(a)$ and $f(b)$. The area of the trapezoid is larger than the area under the curve $y = f(x)$ and hence the lower bound follows in this case with $1/C = 1/2$.

Finally, if $f(a) < 0$ and $f(b) < 0$, then $|h| \ge |f|$ and the lower bound holds with $C = 1$. □

Applying Lemma 18 in the context of strip regions, we have

$$\text{Area}(\mathbb{S} \setminus \Omega_M) + \text{Area}(\Omega_M \setminus \mathbb{S}) < +\infty$$

holds for some M if and only if it holds for all $M > 0$.

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