# Math 427 homework 6 plus

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**Theorem 1.** *Let $f$ be analytic in an open set containing the closed disc $\overline{D}_R(z_0)$. Then $f$ admits a unique power series expansion at $z_0$, say $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$, which converges on $D_{R'}(z_0)$ for some $R' > R$. Then $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{|w-z_0|=R} \frac{f(w)}{(w-z_0)^{n+1}}dw$ for $n \in \mathbb{N}$. In particular, $c_n = \frac{f^{(n)}(z_0)}{n!}$. If $|f(z)| \leq M$ on the boundary of $\overline{D}_R(z_0)$, then $|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$ and $|c_n| \leq \frac{M}{R^n}$ for $n \in \mathbb{N}$.*

This theorem says if $f$ is analytic on a closed disk, then $f(z)$ for any interior point $z$ is completely determined by $f$ restricted to the boundary.

**Problem A** If $f$ has a power series expansion at $z_0$ with radius of convergence $R$, and if $r < R$, then there is a constant $C$ such that $|f(z) - f(z_0)| \leq C|z - z_0|$ provided $|z - z_0| \leq r$.

*proof: cauchy estimates.* Consider $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ on $D_R(z_0)$ for $z \in \overline{D}_r(z_0)$. Choose $s$ such that $r < s < R$. Then $f$ is analytic on $K = \overline{D}_s(z_0)$. Becuase $f$ is continuous, $|f|$ admits a maximum on $K$ (or on $\partial K$). Let $M = \sup_{z\in K}|f(z)|$. By theorem 1, we get $|c_n| \leq \frac{M}{s^n}$ for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
|f(z) - f(z_0)| &= |\sum_{n=1}^{\infty} c_n(z-z_0)^n| \\
&\leq \sum_{n=1}^{\infty} |c_n||z-z_0|^n \\
&= |z-z_0|(\sum_{i=0}^{\infty} |c_{i+1}||z-z_0|^i) \\
&\leq |z-z_0|(\sum_{i=0}^{\infty} \frac{M}{s^{i+1}} r^i) \\
&= \frac{M}{s}|z-z_0|\sum_{i=0}^{\infty}(\frac{r}{s})^i \\
&= \frac{M}{s}\frac{1}{1-\frac{r}{s}}|z-z_0|.
\end{aligned}$$

□

**Remark.** In the proof, $M = \sup_{z\in K}|f(z)| = \sup_{z\in\partial K}|f(z)|$. See Taylor Corollary 3.5.3.

**Problem B** If $f(z) = \sum a_n(z - z_0)$, then $|f(z) - \sum_{n=0}^{k} a_n(z - z_0)^n| \leq D_k|z - z_0|^{k+1}$ where $D_k$ is a constant and $|z - z_0| \leq r < R$.

*Proof.* We use the same idea as in Problem A. Choose some $r < s < R$ and denote $K = \overline{D}_s(z_0)$. Let $M = \sup_{z\in K} |f(z)|$. Then by theorem 1, we get $|c_n| \leq \frac{M}{s^n}$ for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
|f(z) - \sum_{n=0}^{k} a_n(z - z_0)^n| &= |\sum_{n=k+1}^{\infty} c_n(z - z_0)^n| \\
&\leq \sum_{n=k+1}^{\infty} |c_n||z - z_0|^n \\
&= |z - z_0|^{k+1}(\sum_{i=0}^{\infty} |c_{i+k+1}||z - z_0|^i) \\
&\leq |z - z_0|^{k+1}(\sum_{i=0}^{\infty} \frac{M}{s^{i+k+1}} r^i) \\
&= \frac{M}{s^{k+1}} |z - z_0|^{k+1} \sum_{i=0}^{\infty} (\frac{r}{s})^i \\
&= \frac{M}{s^{k+1}} \frac{1}{1 - \frac{r}{s}} |z - z_0|^{k+1}.
\end{aligned}$$

□

*proof 2 for problem A and B.* Let $k \in \mathbb{N}$ be arbitrary. Note that $g_k = \frac{f(z) - \sum_{n=0}^{k} a_n(z-z_0)^n}{(z-z_0)^{k+1}} = \sum_{n=k+1}^{\infty} c_n(z - z_0)^n$ is defined by a power series expanded at $z_0$ with radius of convergence $R$. Indeed, the coefficients of $g_k$ is a k-tail of that of $f$. So the lim sup is the same. Thus $g_k$ is analytic on $K = \overline{D}_s(z_0)$ for some $r < s < R$. It follows $|g_k|$ is uniformly continuous on $K$ and hence attains a maximum $M$. That is, $|\frac{f(z) - \sum_{n=0}^{k} a_n(z-z_0)^n}{(z-z_0)^{k+1}}| \leq M$. □

**Problem C** For large $k$, we can estimate $D_k$ by $1/R^k(R - r)$.

*Proof.* Note $D_k = \frac{M}{s^{k+1}} \frac{1}{1 - \frac{r}{s}} |z - z_0|^{k+1} = \frac{M}{s^k(s-r)}$ for all $r < s < R$. So take $s \to R$. □