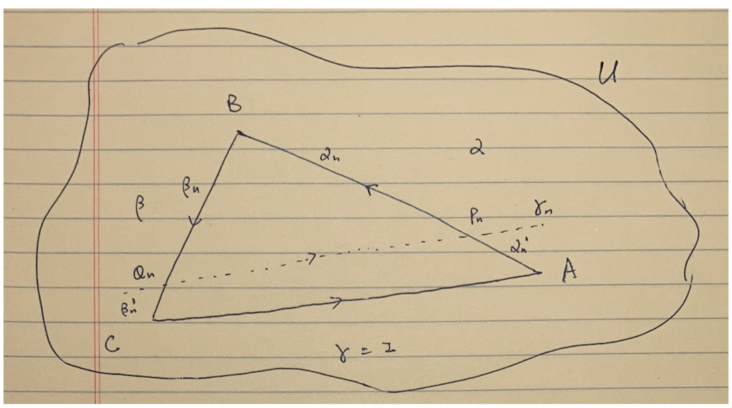
Math 427 homework 2.5.14

Eric Zhang

We give three proofs below. The first is a bit lengthy but with more detail. The second is given by Prof. Marshall. The third clarifies the seemingly mysterious lemma in the exemplary solution.

Proposition 1. *Let U be an open set and Δ be a triangular contained in U. Consider the boundary of Δ to be a closed path with positive orientation, denoted as $\partial\Delta$. Let f be a function continuous on U and analytic on $U \setminus I$ for some interval I being one side of Δ. Then $\int_{\partial\Delta} f = 0$.*

proof 1: analysing the difference. Draw and label the vertices of Δ as A, B, and C with boundary α, β and γ oriented as follows. In particular $\partial\Delta = \alpha + \beta + \gamma$. Without the loss of generality, suppose I coincides with γ. Then consider paths $\gamma_n(t) = \gamma(t) + w_n$ for some fixed $w_n \in \mathbb{C}$ converging to zero, such that γ_n is parallel to γ and intersects α and β at P_n and Q_n respectively. Denote the paths from P_n to B as α_n and the paths from B to Q_n as β_n. Let the triangular region enclosed by α_n, β_n and γ_n be Δ_n. Similarly the boundary $\partial\Delta_n$ is a closed triangular path with positive orientation. In particular $\partial\Delta = \alpha_n + \beta_n + \gamma_n$. Denote $\alpha'_n = \alpha - \alpha_n$ and $\beta'_n = \beta - \beta_n$



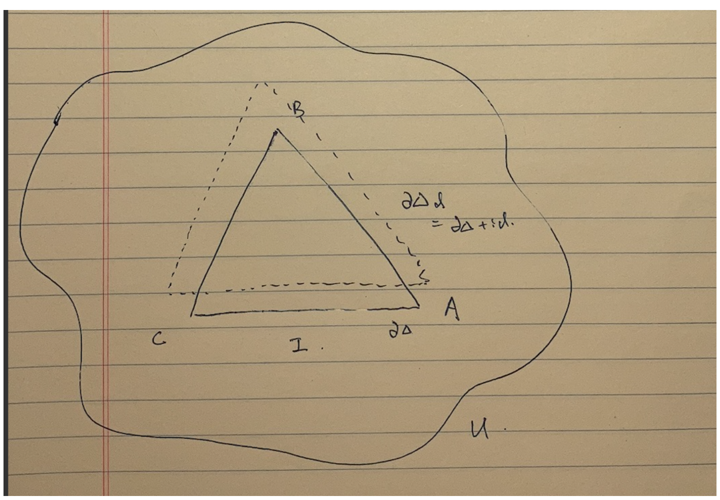
Consider the difference $|\int_{\partial\Delta} f - \int_{\partial\Delta_n} f| = |\int_{\alpha'_n - \gamma_n + \beta'_n + \gamma} f| \leq |\int_{\alpha'_n} f| + |\int_{\beta'_n} f| + |\int_\gamma f - \int_{\gamma_n} f|$. Note $\int_{\alpha_n+\beta_n+\gamma_n} f = 0$ as f is analytic on Δ_n. It suffices to bound this difference by any $\epsilon > 0$.

We use Taylor Theorem 2.4.9 to bound $|\int_{\alpha'_n} f|$ and $|\int_{\beta'_n} f|$ because the length of α'_n and β'_n approaches to 0. Since f is continuous on Δ which is closed and bounded (hence compact in $\mathbb{C}$), there is a maximum $M = \sup_{z\in\Delta} |f(z)|$. It follows that $|\int_{\alpha'_n} f| \leq Ml(\alpha'_n)$ and similarly $|\int_{\beta'_n} f| \leq Ml(\beta'_n)$.

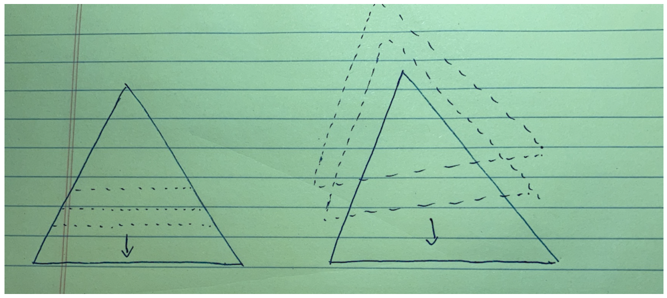
Then we note $\int_\gamma f - \int_{\gamma_n} f = \int_\gamma f(z)dz - \int_{\gamma+w_n} f(w)dw = \int_\gamma f(z)dz - \int_\gamma f(z+w_n)dz$ by a change of variable $w = z+w_n$. It follows that $|\int_\gamma f - \int_{\gamma_n} f| = |\int_\gamma f(z) - f(z+w_n)dz| \leq \int_\gamma |f(z) - f(z+w_n)|dz$. Continuous functions on compact sets are uniformly continuous. So we can find a uniform bound for the integrand $|f(z) - f(z+w_n)|$. With a uniform bound, Theorem 2.4.9 applies to bound $|\int_\gamma f - \int_{\gamma_n} f|$.

Now we put all pieces together. Let $\epsilon > 0$ be arbitrary. There exists N_1 such that $n \geq N_1$ implies $l(\alpha'_n) < \frac{\epsilon}{3M}$ since $\gamma_n \to \gamma$ uniformly. Similarly, there exists N_2 such that $n \geq N_2$ implies $l(\beta'_n) < \frac{\epsilon}{3M}$. Because f is uniformly continuous on Δ, there is $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{3l(\gamma)}$ whenever $|x - y| < \delta$. Then there is N_3 such that $n \geq N_3$ implies $|w_n| < \delta$. It follows when $n \geq N_3$, since $|z + w_n - z| = |w_n| < \delta$, we get $|f(z) - f(z + w_n)| < \frac{\epsilon}{3l(\gamma)}$. Now by Theorem 2.4.9, $|\int_\gamma f - \int_{\gamma_n} f| \leq \int_\gamma |f(z) - f(z+w_n)|dz \leq \frac{\epsilon}{3l(\gamma)} l(\gamma) = \frac{\epsilon}{3}$. Take $N = \max\{N_1, N_2, N_3\}$. For all $n \geq N$, we have $|\int_{\alpha'_n} f| + |\int_{\beta'_n} f| + |\int_\gamma f - \int_{\gamma_n} f| \leq Ml(\alpha_n) + Ml(\beta_n) + \frac{\epsilon}{3} < \epsilon$. □

proof 2: approximation by paths. Let Δ_d be the triangle Δ shifted vertically by d units, where d is positive if B is above I and negative if B is below I. That is, $\Delta_d = \Delta + id$. For sufficiently small d, since Δ_d does not contain I, then f is analytic on Δ_d and hence $\int_{\partial \Delta_d} f = 0$. It follows that $|\int_{\partial\Delta} f(z)dz| = |\int_{\partial\Delta} f(z)dz - \int_{\partial\Delta_d} f(z)dz| = |\int_{\partial\Delta}(f(z) - f(z+id))dz| \leq \int_{\partial\Delta} |f(z) - f(z+id)|dz$. Note f is uniformly continuous on the closed set enclosed by $\partial\Delta_d$ and $\partial\Delta$. Thus, for d sufficiently small, $|f(z) - f(z+id)|$ can be bounded by any $\epsilon > 0$. Then $\int_{\partial\Delta} |f(z) - f(z+id)|dz < \epsilon l(\partial\Delta)$. □



We may notice a pattern. Both proofs involve a sequence of paths converging uniformly to $\partial\Delta$.



So it is natural to wonder if the following statement holds.

Conjecture 1. *Let $\{\gamma_n\}$ be a sequence of smooth paths on $[a,b]$ which uniformly converges to a path γ. Suppose f is a function which is continuous on some open neighbourhood E containing all of γ_n and γ. Then $\lim \int_{\gamma_n} f = \int_\gamma f$.*

Unfortunately this is not true because γ_n' may not converge even if γ_n converges uniformly and are smooth. Consider the following sequences of real functions $f_n = \frac{\sin nx}{\sqrt{n}}$ defined on $\mathbb{R}$ and $g_n = \frac{1}{\sqrt{n}}x^n$ defined on $(0,1]$. Without uniform convergence on the derivative, the integrands of line integrals may not converge uniformly. So we need stronger assumptions.

Lemma 1. *Let $\{\gamma_n\}$ be a sequence of smooth paths on $[a,b]$ that converges pointwise to a function γ where γ_n' uniformly converges to a function g. Suppose f is a function which is continuous on some open neighbourhood E containing all of γ_n and γ. Then $\lim \int_{\gamma_n} f = \int_\gamma f$.*

Proof. Because $\gamma_n' \to g$ uniformly and γ_n smooth, by a classical result in real analysis, γ_n converges uniformly to γ which is differentiable with $\gamma' = g$. This is Rudin Theorem 7.17. Then $\int_\gamma f$ is well-defined. Next we write $\int_{\gamma_n} f = \int_a^b f(\gamma_n(t))\gamma_n'(t)dt$ and $\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt$. It suffices to show the sequence $\{g_n = f(\gamma_n)\gamma_n'\}$ converges uniformly to $g = f(\gamma)\gamma'$. Since $\gamma_n \to \gamma$ uniformly we can find a compact neighbourhood K containing a k-tail γ_{n+k} and γ, on which f will be uniformly continuous. It follows $f(\gamma_{n+k}) \to f(\gamma)$ uniformly on K. It is clear that all of our functions are bounded on K. Therefore the sequence of products $f(\gamma_{n+k})\gamma_{n+k}' \to f(\gamma)\gamma'$ uniformly on K. □

proof 3 of Proposition 1. Note sequences of paths $\partial\Delta_d$ converges uniformly to $\partial\Delta$. Or we may think there are three uniformly convergent sequences of straight-line paths. Because each sequence of paths are parallel lines, their derivatives are constant and are the same, and thus converges uniformly. By lemma 1, $\int_\Delta f = \lim \int_{\partial\Delta_d} f = 0$. □

References

[1] Walter Rudin, *Principles of mathematical analysis*, McGrew Hill, 3rd ed. 1976.

[2] Joseph Taylor *Complex Varibles*, AMS, 2011.