SOLUTIONS

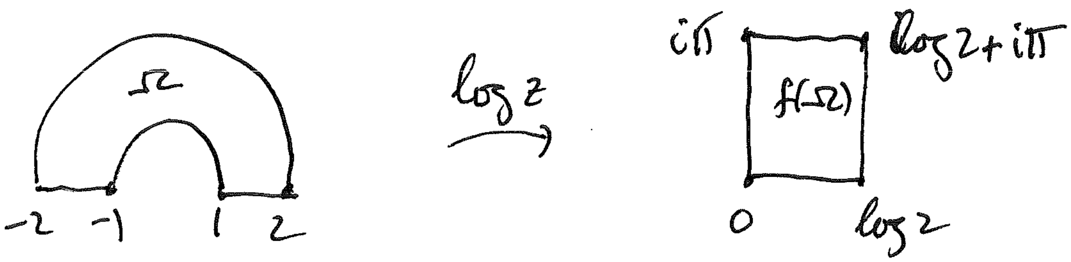
Midterm #2 Math 427 Winter 2023

Name: ______________________________

Student Number: ______________________

Clearly state what theorems or results you are using in each solution. If a theorem has been given a name in class or in the text, you need only state the name of the theorem, but clearly indicate that you have checked all hypotheses.

1. a. Draw a picture of $\Omega = \{z : 1 < |z| < 2 \text{ and } \mathrm{Im} z > 0\}$ and a picture of the image $f(\Omega)$ by the function $\log z$, where $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$. Clearly indicate the boundary of Ω and label 4 points on the boundary. Clearly indicate the boundary of $f(\Omega)$ and the images of your four points.



b. If $\gamma : [0,1] \to \Omega$ is a path in Ω with $\gamma(0) = \gamma(1)$, then let $\sigma : [0,1] \to f(\Omega)$ be a path in $f(\Omega)$ given by $\sigma(t) = \log \gamma(t)$. If g is continuous on Ω, show that

$$\int_\sigma g(e^w)e^w dw = \int_\gamma g(z)dz,$$

by writing each side as an integral over the interval $[0, 1]$.

$$\int_\sigma g(e^w)e^w dw = \int_0^1 g(e^{\log\gamma(t)})\, e^{\log\gamma(t)} \frac{\gamma'(t)}{\gamma(t)} dt = \int_0^1 g(\gamma(t))\gamma'(t)dt$$

$$= \int_\gamma g(z)dz$$

c. Use part b to prove that if g is analytic on Ω then $\int_\gamma g(z)dz = 0$. (Note that Ω is not convex).

THE REGION $f(\Omega)$ IS CONVEX, σ IS A CLOSED PATH AND $g(e^w)e^w$ IS ANALYTIC ON $f(\Omega)$ (PRODUCT & COMPOSITION OF ANALYTIC FUNCTIONS)

BY CAUCHY'S THEOREM, $\int_\sigma g(e^w)e^w dw = 0$.

BY PART b $\int_\gamma g(z)dz = 0$.

2. Let C_1 be the unit circle traced counterclockwise (positive orientation). Compute

$$I = \int_{C_1}\left(\frac{z^2}{e^z}+\frac{e^z}{z^2}\right)dz.$$

$\frac{z^2}{e^z}$ IS ANALYTIC IN $\mathbb{C}$, SO BY CAUCHY'S THEOREM

CONVEX

$$\int_{C_1}\frac{z^2}{e^z}\,dz = 0$$

$$e^w = \int_{C_1}\frac{e^z}{z-w}\,dz/2\pi i$$

FOR $|w|<1$, BY CAUCHY'S INTEGRAL FORMULA SINCE $\mathrm{IND}_{C_1}(w)=1$

$$e^w = \frac{d}{dw}e^w = \int_{C_1}\frac{e^z}{(z-w)^2}\frac{dz}{2\pi i}$$

CIF COROLLARY.

set $w=0$

$$1 = \int_{C_1}\frac{e^z}{z^2}\frac{dz}{2\pi i}$$

THUS $I = 0 + 2\pi i = 2\pi i$

3. Let $C_{3/2}$ denote the circle centered at 0 of radius 3/2 traced counterclockwise (positive orientation). Compute

$$\int_{C_{3/2}} \frac{\sin z}{(z+2)(z-1)} dz.$$

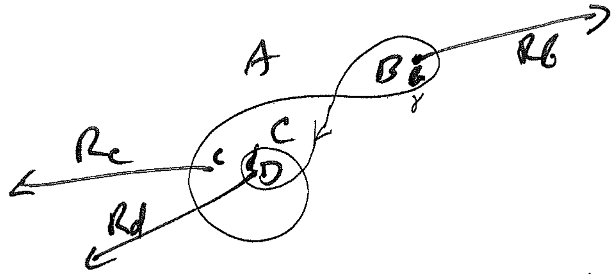
$1 < 3/2 < 2$

So $\frac{\sin z}{z+2} = \frac{e^{iz} - e^{-iz}}{2i(z+2)}$ IS ANALYTIC IN AN OPEN SET CONTAINING $\{z : |z| \leq 3/2\}$.

BY CAUCHY'S INTEGRAL FORMULA $\left[\mathrm{Ind}_{C_{3/2}}(1) = 1\right]$

$$2\pi i \cdot \frac{\sin 1}{1+2} = \int_{C_{3/2}} \frac{\sin z}{z+2} \Big/ z-1 \, dz.$$

4. Find the index or winding number of the following path about each point of its complement. Justify your answers. You may use any result proved in the text or in class, but clearly indicate any result you are using. You may illustrate your answers using additional drawings if it is helpful, but you should at least include a written description of how you computed each of these numbers.



— THE INDEX IS CONSTANT IN EACH COMPONENT OF $\mathbb{C} \setminus \gamma$ AND IS $= 0$ IN THE UNBOUNDED COMPONENT A

CHOOSE POINTS $b \in B$, $c \in C$, AND $d \in D$ AND DRAW RAYS R_b, R_c, R_d FROM b, c, d TO ∞ AS INDICATED

— γ CROSSES R_b ONCE AND $\arg \gamma(t)$ is INCREASING WHERE IT CROSSES SO $\mathrm{Ind}_\gamma(z) = 1$ FOR ALL $z \in B$.

— γ CROSSES R_c ONCE AND ARG $\gamma(t)$ IS DECREASING WHERE IT CROSSES, SO $\mathrm{Ind}_\gamma(z) = -1$ FOR ALL $z \in C$

— γ CROSSES R_d TWICE AND ARG $\gamma(t)$ is DECREASING IN BOTH PLACES, SO $\mathrm{IND}_\gamma(z) = -2$ FOR ALL $z \in D$.

5. Prove that there is a constant $C < \infty$ so that if f is analytic in the unit disk and if $|w| < \frac{1}{2}$ then

$$|f'(w)| \leq C \int_{\frac{3}{4}<|z|<1} |f(x+iy)|dxdy,$$

where $z = x + iy$. Hint: polar coordinates.

BY CAUCHY'S INTEGRAL FORMULA

$3/4 \leq r \leq 1$ $\qquad f(w) = \oint_{|z|=r} \frac{f(z)}{z-w} \frac{dz}{2\pi i}$

AND $f'(w) = \oint_{|z|=r} \frac{f(z)}{(z-w)^2} \frac{dz}{2\pi i}$

IF $|z|=r>3/4$ AND $|w|\leq 1/2$

$$|z-w| \geq |z|-|w| \geq 3/4 - 1/2 = 1/4$$

SO $\quad |f'(w)| \leq \int_0^{2\pi} \frac{|f(re^{i\theta})|}{(1/4)^2} \frac{r\,d\theta}{2\pi} = \frac{8}{\pi} \int_0^{2\pi} |f(re^{i\theta})| r\,d\theta$

INTEGRATE dr:

$$\int_{3/4}^{1} |f'(w)|\,dr \leq \frac{8}{\pi} \int_{3/4}^{1}\int_0^{2\pi} |f(re^{i\theta})| r\,d\theta\,dr$$

SO $\quad |f'(w)| \leq \frac{32}{\pi} \int_{3/4<|z|<1} |f(x+iy)|\,dxdy.$

$C = \frac{32}{\pi}$ WORKS