SOLUTIONS

Midterm #1 Math 427 Winter 2023

Name: ____

Student Number: ____________________________

1. Suppose $z = -1 - \sqrt{3}i$. Then

   a. Re $z =$ -1

   b. Im $z =$ $-\sqrt{3}$

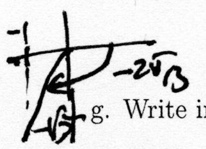
   c. $\overline{z} =$ $-1+\sqrt{3}\,i$

   d. $|z| =$ $\sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = 2$

   e. Write in the form $a+ib$: $(1+i)/z =$

   $$= \frac{1+i}{-1-\sqrt{3}i}\cdot\frac{-1+\sqrt{3}i}{-1+\sqrt{3}i} = \frac{-1-\sqrt{3}+i(-1+\sqrt{3})}{1+3} = -\left(\frac{1+\sqrt{3}}{4}\right)+i\left(\frac{-1+\sqrt{3}}{4}\right)$$

   f. Write z in polar form: $z =$

   $$z = |z|e^{i\arg z} = 2e^{-2\pi i/3}$$

   g. Write in polar form: $z^8 =$

   $$z^8 = 2^8 e^{-16\pi i/3}$$

   h. Let $w = 6e^{i\pi/4}$. Write w in the form $a+ib$, where a, b are real numbers.

   $$= 6\cos\pi/4 + 6i\sin\pi/4 = 3\sqrt{2} + 3\sqrt{2}i$$

   i. Write w/z in the form $a+ib$.

   $$\frac{3\sqrt{2}+3\sqrt{2}i}{-1-\sqrt{3}i} = -3\sqrt{2}\left(\frac{1+i}{1+\sqrt{3}i}\right) = -3\sqrt{2}\,\frac{(1+i)(1-\sqrt{3}i)}{1+3}$$

   $$= -\frac{3\sqrt{2}}{4}\left(1+\sqrt{3}+i(1-\sqrt{3})\right) = \frac{-3\sqrt{2}(1+\sqrt{3})}{4} + i\left[\frac{(1-\sqrt{3})(-1)3\sqrt{2}}{4}\right]$$

   h. Find all solutions ζ to the equation $\zeta^5 = w$.

   If $\zeta = re^{i\theta}$ $\qquad r^5 e^{i5\theta} = 6e^{i\pi/4}$

   $r^5 = 6 \qquad 5\theta = \frac{\pi}{4} + 2\pi k \qquad k \in \mathbb{Z}$

   $r = 6^{1/5} \qquad \theta = \frac{\pi}{20} + \frac{2\pi}{5}k \qquad k = 0,1,2,3,4$ (rest are repeats)

   $$\zeta = 6^{1/5} e^{i\left(\frac{\pi}{20}+\frac{2\pi k}{5}\right)}$$

   $k = 0,1,2,3,4$

2. a. Find the radius of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{z^n}{n3^n}.$$

$$\frac{\left|\frac{z^{n+1}}{(n+1)3^{n+1}}\right|}{\left|\frac{z^n}{n3^n}\right|} = \frac{|z|}{3}\frac{n}{n+1} \xrightarrow[\text{as } n\to\infty]{} \frac{|z|}{3}$$

BY THE RATIO TEST, IF $\frac{|z|}{3} < 1$ THEN THE SERIES CONVERGES ABSOLUTELY AND HENCE CONVERGES. AND BY THE RATIO TEST IF $\frac{|z|}{3} > 1$ THEN THE TERMS OF THE SERIES DO NOT TEND TO 0 SO THE SERIES DIVERGES.

HENCE THE RADIUS OF CONVERGENCE (ABOUT 0) IS 3.

b. What is the largest open set on which this series converges?

$\{z : |z| < 3\}$

c. What is the largest open set on which this series diverges?

$\{z : |z| > 3\}$

3. Expand

$$\frac{z}{(z^2+1)(z^2+2z+2)}$$

in partial fractions where each term in the expansion has a linear function in the denominator. You do not need to simplify the constants. Clearly indicate what you are doing by adding a few words of explanation.

$z^2+1=(z-i)(z+i)$

If $z^2+2z+2=0$: $z=\frac{-2\pm\sqrt{4-8}}{2}=-1\pm i$

So $z^2+2z+2=(z-(-1+i))(z-(-1-i))$

$$\frac{z}{(z-i)(z+i)(z+1-i)(z+1+i)}=\frac{A}{z-i}+\frac{B}{z+i}+\frac{C}{z+1-i}+\frac{D}{z+1+i}$$

MULTIPLY BOTH SIDES BY $z-i$, THEN LET $z\to i$

$\frac{i}{(2i)(1)(1+2i)}=A+0+0+0$ SO $A=\frac{1}{2+4i}$

MULTIPLY BOTH SIDES BY $z+i$, THEN LET $z\to -i$

$\frac{-i}{(-2i)(1-2i)(1)}=B$

SAME FOR $z+1-i$ $\quad z\to -1+i$

$\frac{-1+i}{(-1)(-1+2i)(2i)}=C$

SAME FOR $z+1+i$ $\quad z\to -1-i$

$\frac{-1-i}{(-1-2i)(-1)(-2i)}=D$

4. a. State the definition of a harmonic function on an open set Ω.

A TWICE CONTINUOUSLY DIFFERENTIABLE FUNCTION u DEFINED ON A REGION Ω IS HARMONIC IF

(1) $u_{xx} + u_{yy} = 0$ (u_{xx} & u_{yy} ARE 2ND PARTIAL DERIV.)

AT ALL POINTS OF Ω.

b. Prove that if u is harmonic on an open set Ω, then

$$u_x - iu_y$$

is analytic on Ω, where u_x and u_y are the partial derivatives of $u(x, y)$ with respect to x and y. If you use a theorem to prove this, clearly state the theorem.

Let $\bar{U} = u_x$ AND $\bar{V} = -u_y$ BOTH DIFFERENTIABLE.

$$U_x = u_{xx} \qquad V_y = -u_{yy}$$

By (1) $U_x = V_y$

$$U_y = (u_x)_y \qquad V_x = -u_{yx}$$

So $\bar{U}_y = -\bar{V}_x$

BY THE CAUCHY RIEMANN EQUATIONS

$\bar{U} + i\bar{V}$ is ANALYTIC

$$\| $$

$u_x - iu_y$.

5. Let $f(z) = z + 1/z$. For $r > 0$, let $S_r = \{z : |z| = r\}$. Hint: the polar form of z and $1/z$ are the most useful in the following.

a. Show $f(S_1) = [-2, 2]$

$$z = e^{i\theta} \quad 1/z = e^{-i\theta}$$

$$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

AS θ RANGES FROM $-\pi$ to π, $2\cos\theta$ RANGES FROM -2 to 2 on $\mathbb{R}$.

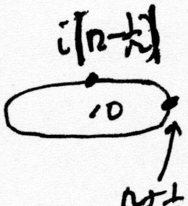
b. If $r \neq 1$ show $f(S_r)$ is an ellipse by writing $f = u + iv$ and showing

$$\left(\frac{u}{r + 1/r}\right)^2 + \left(\frac{v}{r - 1/r}\right)^2 = 1.$$

$$f(re^{i\theta}) = re^{i\theta} + \frac{1}{re^{i\theta}} = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= r\cos\theta + \frac{1}{r}\cos(-\theta) + i\left[r\sin\theta + \frac{1}{r}\sin(-\theta)\right]$$

$$= \underbrace{\left(r + \frac{1}{r}\right)\cos\theta}_{u} + i\underbrace{\left(r - \frac{1}{r}\right)\sin\theta}_{v}$$



$$\left(\frac{u}{r + \frac{1}{r}}\right)^2 + \left(\frac{v}{r - 1/r}\right)^2 = \cos^2\theta + \sin^2\theta = 1$$

$\therefore$ AN ELLIPSE WITH SEMIMAJOR AXIS $r + \frac{1}{r}$, SEMIMINOR AXES $r - \frac{1}{r}$

UNLESS $\theta = \pm\frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$.

c. Show $f(\{z : \arg z = \theta\})$ is a branch of a hyperbola.

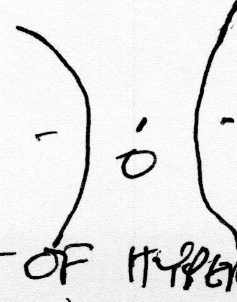
$$f(re^{i\theta}) = \underbrace{\left(r + \frac{1}{r}\right)\cos\theta}_{u} + i\underbrace{\left(r - \frac{1}{r}\right)\sin\theta}_{v}$$

$$\left(\frac{u}{\cos\theta}\right)^2 - \left(\frac{v}{\sin\theta}\right)^2 = \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2$$

$$= r^2 + \frac{1}{r^2} + 2 - \left(r^2 + \frac{1}{r^2} - 2\right)$$

$$= 4$$

equation OF A HYPERBOLA



NOTE: IF $\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$ THEN $\cos\theta > 0$

SO $u > 0$ (GET RIGHT HALF OF HYPERBOLA)

IF $-\pi < \theta < -\frac{\pi}{2}$ OR $\frac{\pi}{2} < \theta < \pi$ THEN $\cos\theta < 0$

SO $u < 0$ SO GET LEFT HALF OF HYPERBOLA