Problem 3.4.12. Let

$$f(z) = \frac{1}{z - z^3}.$$

Analyze each singularity z_0 of f. Is it removable, a pole, or essential? If it is a pole, what is its order? If it is removable, what value should you give the function at z_0 to make it analytic?

Solution. Since the polynomial $z - z^3 = z(1 - z^2) = z(1 - z)(1 + z)$ vanishes at $z = 0, \pm 1$, these points are the singularities of $f(z)$. Each has multiplicity 1 in the polynomial $z - z^3$, so they are poles of order 1.

by example 3.4.10

Problem 3.4.15. Let
$$f(z) = \frac{1}{e^z - 1} - \frac{1}{z}.$$
Analyze each singularity z_0 of f. Is it removable, a pole, or essential? If it is a pole, what is its order? If it is removable, what value should you give the function at z_0 to make it analytic?

Solution. Since $e^z - 1$ is analytic with zero set $Z = \{2\pi ki \mid k \in \mathbb{Z}\}$ and z is analytic with one zero at $0 = 2\pi i \cdot 0$, the singularities of $f(z)$ are precisely the elements of Z. Observe that on $\mathbb{C} \setminus Z$, f can be written as follows:
$$f(z) = \frac{z - e^z + 1}{z(e^z - 1)}.$$
Assuming $k \neq 0$, we see that z is nonzero at $2\pi ki$, and $e^z - 1$ is zero at $2\pi ki$ but its derivative is not. Thus $2\pi ki$ is an order 1 zero for $z(e^z - 1)$. Since $z - e^z + 1$ is nonzero at $2\pi k$, it follows from Example 3.4.10 that $2\pi ki$ is an order 1 pole for $f(z)$.

The singularity at 0 behaves differently. Since 0 is a zero of $z - e^z + 1$, $(z - e^z + 1)' = 1 - e^z$, but not $(z - e^z + 1)'' = -e^z$, we see that 0 is an order 2 zero of $z - e^z + 1$. Moreover, since 0 is a zero of $z(e^z - 1)$, $(z(e^z - 1))' = e^z - 1 + ze^z$, but not $(z(e^z - 1))'' = 2e^z + ze^z$, we see that 0 is an order 2 zero of $z(e^z - 1)$ as well. By Example 3.4.10, 0 is a removable zero of $f(z)$. To make $f(z)$ analytic near 0, we can set
$$f(0) := \lim_{z \to 0} \frac{z - e^z + 1}{z(e^z - 1)} = \lim_{z \to 0} \frac{-e^z}{2e^z + ze^z} = -\frac{1}{2}.$$
Theorem 3.4.13 (a) also applies

One intuitive but maybe not rigorous approach would be

to expand $\frac{1}{e^z-1}$ as Laurent series.

$$\frac{1}{e^z-1} = \frac{1}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots} = \frac{1}{z} \cdot \frac{1}{1 + \left(\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)} = \frac{1}{z} \cdot \frac{1}{1+A}$$

$$= \frac{1}{z} \cdot \left(1 - A + A^2 - A^3 \cdots\right)$$ if $|A| < 1$ (this is true near $z=0$)

$A = \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots$ implies $A^i = \left(\frac{z}{2!}\right)^i +$ higher degree terms

So $$\frac{1}{e^z-1} = \frac{1}{z} \cdot 1 - \frac{1}{z} \cdot A + \frac{1}{z} \cdot A^2 - \cdots$$

$$= \frac{1}{z} - \left(\frac{1}{2} + \text{positive degree terms}\right) + \left(\text{positive degree terms}\right)$$

it follows $\frac{1}{e^z-1} - \frac{1}{z}$ does not have a simple pole at $z=0$

since the -1 degree term cancels out.

Problem 3.4.16. Let

$$f(z) = \frac{\log z}{(1-z)^2}, \quad = \frac{\log z}{(z-1)^2}$$

where log is the principal branch of the log function. Analyze each singularity z_0 of f. Is it removable, a pole, or essential? If it is a pole, what is its order? If it is removable, what value should you give the function at z_0 to make it analytic?

Solution. The function $f(z)$ is analytic on $\mathbb{C} \setminus ((-\infty, 0] \cup \{1\})$, so the only isolated singularity of $f(z)$ is 1. It is clearly a zero of degree 2 of $(1-z)^2$, and since $(\log z)' = 1/z$ is nonzero at 1, it is a order 1 zero of $\log z$. By Example 3.4.10, 1 is an order $2 - 1 = 1$ pole of $f(z)$.

again. we may consider the laurent series of f at $z=1$

or we consider the power series of $\log z$ at $z=1$

For the principal branch of $\log z$. a power series at $z=1$ exists because locally it is analytic around a small neighbourhood

$$\log z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n$$

this shows $\log z$ at a order one zero at $z=1$

or it implies $f(z) = \frac{1}{(1-z)}$ + higher degree terms.

so $f(z)$ has a order one pole at $z=1$.

Problem 3.5.3. Find where the function $(z-1)^2$ attains its maximum modulus on the triangle with vertices at $0, 1+i, 1-i$.

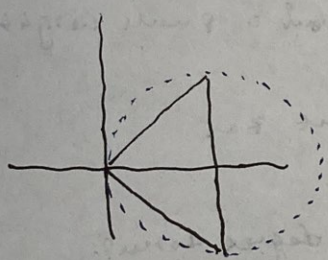
Solution. Let $f(z) = (z-1)^2$. Let Δ be the triangle described in the problem statement, and note that f is continuous on Δ. Since Δ° is a connected and bounded open subset of $\mathbb{C}$ on which f is analytic, the maximum modulus of f is attained only on $\partial\Delta^\circ = \partial\Delta$. In other words, the points at which $|f(z)|$ is maximized belong to the following line segments: $L_1 = \{1+(2t-1)i \mid 0 \le t \le 1\}$, $L_2 = \{t(1+i) \mid 0 \le t \le 1\}$, $L_3 = \{t(1-i) \mid 0 \le t \le 1\}$. So consider the functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ given by

$$f_1(t) = |f(1+(2t-1)i)| = |(2t-1)i|^2 = (2t-1)^2,$$
$$f_2(t) = |f(t(1+i))| = |ti+t-1|^2 = t^2 + (t-1)^2.$$

just parametrize f along the boundary like in Math 126 the rest is single variable calculus.

Since $f_1'(t) = 8t-4$ is negative on $[0,1/2]$ and positive on $[1/2,1]$, $|f(z)|$ is maximized on L_1 only at one or both of the endpoints $1-i, 1+i$. Next, since $f_2'(t) = 2t + 2(t-1) = 4t-2$, we see that f_2 is decreasing on $[0,1/2]$ and increasing on $[1/2,1]$, so $|f(z)|$ is maximized on L_2 possibly at one or both of the endpoints 0, $1+i$. In fact, $f_2(t) = |ti+t-1|^2 = |-ti+t-1|^2 = |f(t(1-i))|$ as well, so $|f(z)|$ is maximized on L_3 at 0 or $1-i$. Noting that $|f(0)| = |f(1+i)| = |f(1-i)| = 1$, we deduce that the maximum modulus of f is 1 and it is achieved on the points $0, 1+i, 1-i$.

another solution due to another student, slightly adjusted,



embed the triangle Δ into the circle $|z-1| = 1$ and maximal modulus principle tells us the maximum of $|(z-1)^2|$ is attached at $\partial\{|z-1|=1\}$ which only the vertex satisfies.

everything works out because $(z-1)^2$ is entire so no worries on the assumptions of maximal modulus principle.

Problem 3.5.5. Show that if f is a non-constant, continuous function on $\overline{D}_1(0)$, which is analytic on $D_1(0)$ and $|f(z)| = 1$ for all z on the unit circle, then f has a zero somewhere in $D_1(0)$.

Solution. Suppose that f is continuous on $\overline{D}_1(0)$, analytic on $D_1(0)$, and satisfies $|f(z)| = 1$ on the unit circle. Assuming that f has no zeros in $D_1(0)$, we will prove that f is constant.

By Corollary 3.5.3, we have $|f(z)| < 1$ for all $z \in D_1(0)$. Since f has no zeros on $D_1(0)$ or the unit circle, the function $1/f$ is continuous on $\overline{D}_1(0)$ and analytic on $D_1(0)$. Moreover, $1/|f(z)| = 1$ for all $|z| = 1$ and $1/|f(z)| > 1$ for all $z \in D_1(0)$. Therefore, since $1/|f(z)|$ is a continuous real-valued function on the compact set $\overline{D}_1(0)$, it achieves a maximum value, which is necessarily at a point in $D_1(0)$ by our previous observation. By the maximum modulus theorem, $1/f$ is constant on $D_1(0)$, so f is constant on $D_1(0)$ as well. Then, by continuity, f is in fact constant on all of $\overline{D}_1(0)$.

because $|1/f| > 1$ in $D_1(0)$

Similarly. assume f has no zero in $D_1(0)$

then $1/f$ is well defined on $\overline{D}_1(0)$ (we know $f \neq 0$ on $S^1 = \partial D_1(0)$)

$1/f$ is cts on $\overline{D}_1(0)$, analytic on $D_1(0)$ because f is such.

then by Corollary 3.5.3. $1/f$ attains maximum on $\partial D_1(0)$

but it is false and hence $1/f$ is a constant

which means f is constant and it is a contradiction.

In the following problems, let $D = \{z : |z| < 1\}$ be the unit disk.

Problem A.

A1. If c and a are constants with $|c| = 1$ and $|a| < 1$, prove that

$$\varphi(z) = c\left(\frac{z-a}{1-\bar{a}z}\right) \tag{1}$$

is a one-to-one analytic map of D onto D. Hint: explicitly find the inverse function.

A2. Show that φ^{-1} is of the same form as φ, but with different constants.

A3. Show that $|\varphi(z)| = 1$ when $|z| = 1$.

Solution. Clearly, φ is analytic on all of $\mathbb{C}$ if $\bar{a} = 0$. If $\bar{a} \neq 0$, then it is analytic everywhere except $1/\bar{a}$, but since $1/|\bar{a}| = 1/|a| > 1$, φ is still analytic on D. Thus φ is guaranteed analytic on D. Furthermore, if $|z| = 1$, then $\bar{z} = z^{-1}$ and so

$$\begin{aligned}
|\varphi(z)| &= \left|c\frac{z-a}{1-\bar{a}z}\right| \\
&= \left|\frac{z-a}{z(\bar{z}-\bar{a})}\right| \\
&= \left|\frac{z-a}{\bar{z}-\bar{a}}\right| \\
&= 1.
\end{aligned}$$

~~also you want to define φ on D̄~~

why φ is continuous on $\bar{D}$?

$1 - \bar{a}z \neq 0$ on $\bar{D}$

since $1/|\bar{a}| \notin \bar{D} \Rightarrow \varphi$ cts on $\bar{D}$

~~and hence~~

This proves A3, and by Corollary 3.5.3, $|\varphi(z)| < 1$ for all $|z| < 1$. In other words, φ maps D to D.

?

~~so φ is continuous~~

Now, define a function $\tilde{\varphi} : D \to D$ by

$$\tilde{\varphi}(z) = \bar{c}\left(\frac{z+ac}{1+\bar{a}cz}\right).$$

or $\dfrac{z+ca}{c+a\bar{z}}$

Since $|-ac| = |a|$, we see by the same reasoning as before that $\tilde{\varphi}$ is analytic on D and maps D to D. Thus $\tilde{\varphi} \circ \varphi$ maps D to D, and for each $z \in D$,

$$\begin{aligned}
\tilde{\varphi} \circ \varphi(z) &= \bar{c}\left(\frac{c(z-a)/(1-\bar{a}z) + ac}{1 + \bar{a}\bar{c}c(z-a)/(1-\bar{a}\bar{c}z)}\right) \\
&= \frac{(z-a)/(1-\bar{a}z) + a}{1 + \bar{a}(z-a)/(1-\bar{a}z)} \\
&= \frac{z - a + a - |a|^2 z}{1 - \bar{a}z + \bar{a}z - |a|^2} \\
&= \frac{z - |a|^2 z}{1 - |a|^2} \\
&= z,
\end{aligned}$$

and by symmetry, $\varphi \circ \tilde{\varphi}(z) = z$ as well. Hence φ is a bijection with inverse $\tilde{\varphi}$. This proves A1 and A2.

Problem D. Suppose f is analytic in D and suppose $|f(z)| < 1$ in D. Let $a = f(0)$. Show that $f(z) \neq 0$ if $|z| < |a|$. Hint: Use f to build another function g with $g(0) = 0$ and $|g| < 1$ on D.

Solution. Let $\varphi : D \to D$ be the analytic function

$$\varphi(z) = \frac{z - a}{1 - \bar{a}z}$$

from Problem A, and consider the composition $g = \varphi \circ f$, which is well-defined since $f(D) \subseteq D$. We clearly have $g(0) = 0$, so Schwartz's Lemma implies $|g(z)| \leq |z|$ for all $z \in D$. Thus if $z \in D$ is such that $f(z) = 0$, we have

$$|a| = |-a| = |g(z)| \leq |z|.$$

This proves the contrapositive of our desired result, so we are done.