MATH 427 - Homework 7

§3.3: Liouville's Theorem

4. **Proposition.** If f is an entire function satisfying $|f(z)| \geq 1$ on $\mathbb{C}$, then f is constant.

why we need $f(z) \neq 0$ is that we want $1/z$ to be defined everywhere in $\mathbb{C}$

Proof. Since $|f(z)| \geq 1$ on $\mathbb{C}$, $f(z) \neq 0$, and so $f(z)$ entire implies $1/f(z)$ entire. Additionally, $|1/f(z)| = 1/|f(z)| \leq 1$ for $z \in \mathbb{C}$, so that $1/f(z)$ is a bounded entire function. Then by Liouville's theorem, $1/f(z)$ is a constant function, which implies $f(z)$ constant as well. □

5. **Proposition.** If an entire function has real part which is bounded above, then it is constant.

Proof. Let f be an entire function, and suppose that $\mathrm{Re}(f(z)) \leq M$ for some $M \in \mathbb{R}$ and all $z \in \mathbb{C}$. Consider the function g defined $g(z) = f(z) - (M+1)$. Since both f and the constant function $M+1$ are entire, g is also entire. Then for any $z \in \mathbb{C}$,

$$\mathrm{Re}(g(z)) = \mathrm{Re}(f(z) - (M+1)) = \mathrm{Re}(f(z)) - M - 1 \leq M - M - 1 = -1,$$

nice.

so that by Thm. 1.1.8,

$$|g(z)| \geq |\mathrm{Re}(g(z))| \geq 1.$$

By the previous question, g must be constant, and so f is as well. □

6. **Proposition.** If an entire function f is not constant, then its range $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Proof. We prove the contrapositive. Suppose $f(\mathbb{C})$ is not dense in $\mathbb{C}$, so that there exists $z_0 \in \mathbb{C}$ and $r > 0$ such that $D_r(z_0) \cap f(\mathbb{C}) = \emptyset$. In other words, $|f(z) - z_0| \geq r$ for all $z \in \mathbb{C}$. Divide through by $r > 0$ to get

good.

$$\frac{1}{r} \cdot |f(z) - z_0| = \left|\frac{f(z) - z_0}{r}\right| \geq 1.$$

Since f and the constant functions $1/r$ and z_0 are entire, by Thm. 2.2.6, $(f(z) - z_0)/r$ is entire. So by Q. 3.3.4, $(f(z) - z_0)/r$ is a constant function. This implies f constant as well. □

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16. **Proposition.** If an entire function f satisfies an inequality of the form

$$|f(z)| \leq A + B\log|z| \quad \text{for all } z \in \mathbb{C} \text{ with } |z| \geq 1,$$

where A and B are positive constants, then f is constant.

Proof. Let $R > 1$. Since f is entire, it is analytic on the open set $D_{R+1}(0)$ containing $\overline{D}_R(0)$. On the boundary of $\overline{D}_R(0)$, $|z| = R$, and so

$$|f(z)| \leq A + B\log|z| = A + B\log R.$$

Let n be any positive integer. By Thm. 3.2.9,

$$|f^{(n)}(0)| \leq \frac{n!(A + B\log R)}{R^n}.$$

Since this holds for arbitrarily large R, taking the limit $R \to \infty$ gives us $|f^{(n)}(0)| = 0$. (Note that for large R, $R^n \gg \log R$, or use L'Hôpital's). By Cor. 3.2.4,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = f(0) + \sum_{n=1}^{\infty} \frac{0}{n!} z^n = f(0),$$

and so f must be a constant function. □

or we can consider the power series expansion of f at $z=$ (this exists because f is analytic over F. Then note for any open disk $D_R(0)$. the coefs of power series is bounded by M/R^n which goes to zero

you used $f^{(n)}(0)$; I used C_n. same thing

§3.4: Zeroes and Singularities

3. **Proposition.** If f is an entire function such that $f(1/n) = 0$ for all integers $n \geq 1$, then in fact $f(z) = 0$ for all $z \in \mathbb{C}$.

Proof. Since $0 \in \mathbb{C}$ is a limit point of $S := \{1/n : n \in \mathbb{N}\}$, S is not a discrete subset of $\mathbb{C}$. Let $g(z) = 0$ for all $z \in \mathbb{C}$. Since g is a constant function, it is entire, and we are given f entire as well. Then by Thm. 3.4.4, $f(w) = g(w)$ for $w \in S$ implies $f(z) = g(z) = 0$ for all $z \in \mathbb{C}$. □

Proposition. The function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined $f(z) = \cos(2\pi/z) - 1$ is a non-constant analytic function on $\mathbb{C} \setminus \{0\}$ such that $f(1/n) = 0$ for all integers $n \geq 1$.

Proof. By Thm. 2.2.6, the function $1/z$ is differentiable whenever $z \neq 0$. Additionally, the constant functions 2π and 1, as well as $\cos z = (e^{iz} + e^{-iz})/2$ are all entire. So by Thm. 2.2.6 and 2.2.7, f is analytic on $\mathbb{C} \setminus \{0\}$. For any integer $n \geq 1$,

$$f(1/n) = \cos\left(\frac{2\pi}{1/n}\right) - 1 = \cos(2\pi n) - 1 = 1 - 1 = 0.$$

or $g(z) = \sin\left(\frac{2\pi}{z}\right)$

However, f is not identically zero; for instance,

$$f(4) = \cos(2\pi/4) - 1 = \cos(\pi/2) - 1 = -1 \neq 0.$$

□

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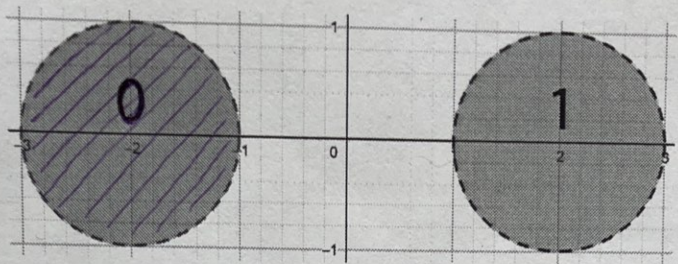
6. **Proposition.** Theorem 3.4.2 may not hold if U is not connected.

Proof. Let $U = D_1(-2) \cup D_1(2)$, and consider $f : U \to \mathbb{C}$ defined $f(z) = 0$ for $z \in D_1(-2)$ and $f(z) = 1$ for $z \in D_1(2)$. Clearly, f is analytic on U and not identically zero. However, each $z \in D_1(-2)$ is a zero of f, and the set $D_1(-2)$ is uncountable. $\square$



9. **Proposition.** Let f, g be analytic functions on an open set U, $z_0 \in U$, and $f(z_0) = g(z_0) = 0$. Suppose also that g is not identically zero on an open disc centered at z_0. Then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)},$$

and this limit always exists if $\pm\infty$ is allowed as possible values.

Proof. Suppose that f is not identically zero on an open disc centered at z_0. Then by Thm. 3.4.1, for some $m, n \geq 0$, we can write $f(z) = (z - z_0)^m F(z)$ and $g(z) = (z - z_0)^n G(z)$ on a disc $D_r(z_0)$, where $F(z)$ and $G(z)$ are analytic, and $F(z), G(z) \neq 0$ on the disc. But in fact m, n are strictly positive, since we are given $f(z_0) = g(z_0) = 0$. On the left hand side,

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{(z-z_0)^m F(z)}{(z-z_0)^n G(z)} = \frac{F(z_0)}{G(z_0)} \left[\lim_{z\to z_0} \frac{(z-z_0)^m}{(z-z_0)^n} \right].$$

If $m = n$, then the limit in square brackets converges to 1, if $m > n$, then it converges to 0, and if $m < n$, then it blows up to $\pm\infty$. On the right hand side,

$$\lim_{z\to z_0} \frac{f'(z)}{g'(z)} = \lim_{z\to z_0} \frac{m(z-z_0)^{m-1}F(z) + (z-z_0)^m F'(z)}{n(z-z_0)^{n-1}G(z) + (z-z_0)^n G'(z)} =$$

$$\lim_{z\to z_0} \frac{F(z) + (z-z_0)F'(z)/m}{G(z) + (z-z_0)G'(z)/n} \left[\lim_{z\to z_0} \frac{m(z-z_0)^{m-1}}{n(z-z_0)^{n-1}} \right] = \frac{F(z_0)}{G(z_0)} \left[\lim_{z\to z_0} \frac{m(z-z_0)^{m-1}}{n(z-z_0)^{n-1}} \right].$$

Note that $F'(z_0)$ is a constant and F' is continuous, so $(z - z_0)F'(z)/m$ vanishes as $z \to z_0$ (same goes for G'). If $m = n$, then the limit in the square brackets converges to 1, if $m > n$, then it converges to 0, and if $m < n$, then it blows up to $\pm\infty$. Additionally, it also has the same constant multiplier $F(z_0)/G(z_0)$. So behaves in precisely the same way as the the left hand limit. If f is identically zero on an open disc centered at z_0, then it is clear that both limits converge to zero. Therefore the limits always exist and are equal. $\square$

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11. **Proposition.** If f is a function which is analytic in the exterior of the closed disc $\overline{D}_r(0)$, and if $\lim_{z\to\infty} f(z) = 0$, then f has a power series expansion of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^{-n}$$

converging on the set $\{z \in \mathbb{C} : |z| > r\}$.

Proof. Let $D = \{z \in \mathbb{C} : 0 < |z| < 1/r\}$, and define $g : D \to \mathbb{C}$ by $g(z) = f(1/z)$. If $z \in D$, $z \neq 0$, so the function $1/z$ is analytic on D. Additionally, $z \in D$ implies $|1/z| > r$, so f is differentiable at $1/z$. Then by Thm. 2.2.7, g is analytic on D. Since

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} f(1/z) = \lim_{z\to\infty} f(z) = 0,$$

by Thm. 3.4.13(a), g has a removable singularity at 0. So the function $h : D_{1/r}(0) \to \mathbb{C}$ defined $h(0) = 0$ and $h(z) = g(z)$ for $z \in D$ is analytic on its domain. Then we can write

$$h(z) = \sum_{n=1}^{\infty} a_n z^n$$

as a power series converging on $D_{1/r}(0)$ (note the constant term is 0 since $h(0) = 0$). Let $S = \{z \in \mathbb{C} : |z| > r\}$. For $z \in S$, $0 < |1/z| < 1/r$ lies in $D \subseteq D_{1/r}(0)$, and so

$$f(z) = g(1/z) = h(1/z) = \sum_{n=1}^{\infty} a_n z^{-n}$$

converges, as desired. $\square$

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