§3.1: Uniform Convergence

5

6. **Proposition.** For each $r > 0$, the series

$$S = \sum_{k=0}^{\infty} \frac{1}{k^2 - z}$$

converges uniformly on the set

$$E_r = \{z : |z| \leq r, z \neq k^2 \text{ for } k = 0, 1, 2, \ldots\}$$

Proof. Let $r > 0$. If $z \in E_r$, then $1/(k^2 - z)$ is well-defined since $k^2 - z \neq 0$. Pick $n \in \mathbb{N}$ large enough so that $n^2 > 2r$. Then for $k \geq n$,

$$\left|\frac{1}{k^2 - z}\right| = \frac{1}{|k^2 - z|} \leq \frac{1}{|k^2 - |z||} = \frac{1}{k^2 - |z|} \leq \frac{1}{k^2 - r} \leq \frac{1}{k^2 - (k^2/2)} = \frac{2}{k^2},$$

where the first inequality follows from the reverse triangle inequality, and the other steps from $k^2 \geq n^2 > 2r \geq 2|z|$. Since $\sum_{k=1}^{\infty} 2/k^2$ is a constant multiple of the p-series for $p = 2 > 1$, it converges. Note that we can ignore the first n terms of S without affecting its convergence. So by the Weierstrass M-test, S converges uniformly on E_r. □

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7. **Proposition.** The series

$$S = \sum_{k=1}^{\infty} k^{-z}$$

converges uniformly on each set of the form $E_s = \{z \in \mathbb{C} : \text{Re}(z) > s\}$, with $s > 1$.

Proof. Let $s > 1$, and let $z = x + yi \in E_s$, so that $x > s$. Then

$$|k^{-z}| = |e^{-z \log k}| = e^{\text{Re}(-z \log k)} = e^{-x \log k} = k^{-x} \leq k^{-s},$$

where we've used $k \neq 0$, Thm. 1.4.8(a), and Thm. 1.3.7(b). Since $\sum_{k=1}^{\infty} k^{-s}$ is the p-series for $p = s > 1$, it converges. So by the Weierstrass M-test, S converges uniformly on E_s. □

You can just say

$$|k^{-z}| = |e^{-z \log k}| = e^{\text{Re}(-z \log k)}$$

$$= e^{\log k \ \text{Re}(-z)} = k^{-\text{Re}(z)} \leq k^{-s} \quad \text{for } \text{Re}(z) > s > 1$$

without writing down the cartesian form for z.

We can check that

$\limsup_{k\to\infty} \sqrt[k]{\left|\frac{(-1)^k}{k+1}\right|} = \limsup_{k\to\infty} e^{-\frac{\ln(k+1)}{k}} = e^{\limsup_{k\to\infty} -\frac{\ln(k+1)}{k}}$

3

15. Proposition. The power series expansion about 0 of

$$\int_0^z \frac{1}{1+w}\,dw = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} z^{k+1}$$

has radius of convergence $R = 1$, and converges to $\log(1+z)$.

Proof. From the geometric series formula, we have

$$\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k$$

$= e^0 = 1$. which implies ~~is~~ the radius of convergence for the new power series

if $|z| < 1$, and divergence for $|z| > 1$. So its radius of convergence is 1. By Thm. 3.1.11,

I am not sure if theorem 3.1.11 says about radius of convergence of the new power series

$$\int_0^z \frac{1}{1+w}\,dw = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} z^{k+1}$$

has the same radius of convergence $R = 1$.? By Example 2.2.11 and the chain rule, we have $\log(1+w)' = 1/(1+w)$, so $\log(1+w)$ is the antiderivative of $1/(1+w)$. By the FTC, ?

$$\int_0^z \frac{1}{1+w}\,dw = \log(1+z) - \log(1-0) = \log(1+z).$$ ∎

Let Log z be principal branch, then log $1+w$ is analytic on: $1+w \notin (-\infty, 0]$ i.e. $w \notin (-\infty, -1]$ Note $D_0(1) \cap (-\infty,-1] = \emptyset$ so log $1+w$ indeed has derivative on $D_0(1)$ which is $1/1+w$. So FTC may apply.

§3.2: Power Series Expansions

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6. Proposition. The power series for $1/\cos z$ about $z_0 = 0$ has radius of convergence $\pi/2$.

Proof. The function $1/\cos z$ is analytic on $U = \{z \in \mathbb{C} : \cos z \neq 0\}$. We have $\cos z = 0$ when

$$e^{iz} = -e^{-iz} \iff e^{2iz} = -1 \iff 2z = \pi + 2\pi k \iff z = \pi/2 + \pi k, \quad \text{where } k \in \mathbb{Z}.$$

So the largest disc about z_0 contained in U is $D_{\pi/2}(0)$. By Thm. 3.2.5, the power series for $1/\cos z$ converges on $D_{\pi/2}(0)$. Surely it doesn't converge on $D_r(0)$ for any $r > \pi/2$, since $1/\cos z$ is not even well-defined on $D_r(0)$. So $\pi/2$ is its radius of convergence. ∎

4

8. Proposition. The function f defined by $(\sin z)/z$ when $z \neq 0$ and 1 when $z = 0$ is analytic on the whole complex plane.

Proof. The power series expansion for ~~$(\sin z)/z$~~ $\sin(z)$ is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

why it converges on $\mathbb{C}$? consider $(2k+1)! < k^k$

and converges on all $\mathbb{C}$. For $z \neq 0$, we can divide through by z to get

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots \quad \cancel{\text{for } z \neq 0.} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!}$$

At $z = 0$, this power series is equal to 1. By our definition of f,

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots \quad \text{for } z \in \mathbb{C}.$$

So f has a power series converging on $\mathbb{C}$,? and is therefore analytic everywhere. ∎

I may say consider $\lim_{w\to 0} \frac{\frac{\sin w}{w} - 1}{w - 0} = \lim_{w\to 0} \frac{\sum_{k=1}^{\infty} (-1)^k \frac{w^{2k}}{(2k+1)}}{w - 0} = 0$

since $\frac{\sin w}{w} - 1 = -\frac{w^2}{3} + \frac{w^4}{5!} \cdots \to 0$ as $w \to 0$. So $\frac{\sin z}{z}$ is analytic at 0.

the series converges absolutely so we may rearrange the sum.

4 9. **Proposition.** If f is analytic and not identically 0 on a disc $D_r(z_0)$, then there exists some $k \in \mathbb{Z}_{\geq 0}$ and a function g analytic on $D_r(z_0)$ such that $f(z) = (z - z_0)^k g(z)$ and $g(z_0) \neq 0$.

Proof. Since f is analytic on $D_r(z_0)$, it has power series

$g(z) = \sum_{n=0}^{\infty} c_{k+n}(z-z_0)^n$ converges on $D_r(z_0)$

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

NOT all power series is analytic. It must converge first to be a well-define function!

converging on $D_r(z_0)$. Since f is not the zero function, there exists a smallest index $k \in \mathbb{Z}_{\geq 0}$ such that $c_k \neq 0$. This implies $c_1 = \cdots = c_{k-1} = 0$, and so we can write

$$f(z) = \sum_{n=k}^{\infty} c_n (z - z_0)^n = (z - z_0)^k \left[\sum_{n=0}^{\infty} c_{k+n}(z - z_0)^n \right].$$

because $\limsup |c_{k+n}|^{1/n} = \lim\sup |c_n|^{1/n}$ as sequence $\{c_{k+n}\}$ is just a tail of $\{c_n\}$

Then set $g(z)$ equal to the series in brackets, and note $f(z) = (z - z_0)^k g(z)$ and $g(z_0) \neq 0$, since $c_k \neq 0$. Since g is given as a power series, it is analytic. ? □

5 10. **Proposition.** If f is analytic on $D_R(z_0)$ and $|f(z)| \leq M$ on $D_R(z_0)$, then $|f'(z_0)| \leq M/R$.

Proof. Let $0 < r < R$, so that $\overline{D}_r(z_0) \subseteq D_R(z_0)$. Then since $|f(z)| \leq M$ on $D_R(z_0)$, and in particular, the boundary of $D_r(z_0)$, by Thm. 3.2.9, $|f'(z_0)| \leq M/r$. Since this is true of all $r < R$, it must be that $|f'(z_0)| \leq M/R$. (taking $r \to R$) □

14. **Proposition.** If f is continuous on an open set U and analytic on $U \setminus E$, where E is either a point or a line segment, then in fact f is analytic on all of U.

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Proof. Let Δ be a triangle contained in U, and let $\partial\Delta$ denote its boundary. By Thm. 2.5.9 and Q. 2.5.14, despite f possibly being non-analytic on a point or a line, we still have

$$\int_{\partial\Delta} f(z)\, dz = 0.$$

So by Morera's theorem, f must actually be analytic on all of U. □

A. **Proposition.** If f has a power series expansion at z_0 with radius of convergence R, and if $r < R$, then there is a constant C so that $|f(z) - f(z_0)| \le C|z - z_0|$ provided $|z - z_0| \le r$.

5

Proof. Since f has a power series expansion on $D_R(z_0)$, by Thm. 3.2.1, it has an analytic derivative f'. In particular, f' is continuous on the compact set $\overline{D}_r(z_0)$, and since a continuous function on a compact set is bounded, we can pick C such that $|f'(z)| \le C$ for all $z \in \overline{D}_r(z_0)$. Since $\overline{D}_r(z_0)$ is a convex set, the line from z_0 to z is contained in $\overline{D}_r(z_0)$, and so by the FTC and Thm. 2.4.9,

$$|f(z) - f(z_0)| = \left| \int_{z_0}^{z} f'(w)\, dw \right| \le C|z - z_0|,$$

where $|z - z_0|$ is the length of the line between z and z_0. □

B. **Proposition.** If $f(z) = \sum a_n (z - z_0)^n$, then

$$\left| f(z) - \sum_{n=0}^{k} a_n (z - z_0)^n \right| \le D_k |z - z_0|^{k+1},$$

where D_k is a constant and $|z - z_0| \le r < R$.

Proof. Let part A be our base case, and suppose for induction we have an estimate like above for any function with a power series estimated up to degree $k - 1$. In particular, for

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1},$$

we have

$$\left| f'(z) - \sum_{n=1}^{k} n a_n (z - z_0)^{n-1} \right| \le D_{k-1} |z - z_0|^k$$

for some constant D_{k-1}. As in the previous part, take the line integral from z_0 to z to get

$$\left| \int_{z_0}^{z} \left(f'(w) - \sum_{n=1}^{k} n a_n (w - z_0)^{n-1} \right) dw \right| = \left| f(z) - f(z_0) - \sum_{n=1}^{k} a_n \left\{ (z - z_0)^n - (z_0 - z_0)^n \right\} \right|$$

$$= \left| f(z) - \sum_{n=0}^{k} a_n (z - z_0)^n \right|,$$

$\left|\int_{z_0}^{z} \left\{ f'(w) - \sum_{n=1}^{k} n a_n (w - z_0)^{n-1} \right\} dw \right|$

triangle inequality

where in the last step, we used $f(z_0) = a_0$. Parameterize the line by $\gamma(t) = (z - z_0)t + z_0$ with $t \in (0, 1)$. Then the expression above is bounded by

$\le \int_{z_0}^{z} D_{k-1} |w - z_0|^k \, dw$

by 2.4.9

$\le \max_{\overline{D}_{z_0}(r)} \left\{ D_{k-1} |w - z_0|^k \right\} \cdot |z - z_0|$

$= \max_{\overline{D}_{z_0}(r)} D_{k-1} |z - z_0|^k \cdot \frac{|w - z_0|^k}{|z - z_0|^k} \cdot |z - z_0|$

$= D_{k-1} |z - z_0|^{k+1} \cdot \max_{\overline{D}_{z_0}(r)} \frac{|w - z_0|^k}{|z - z_0|^k}$

$$\int_\gamma D_{k-1} |w - z_0|^k \, |dw| = \int_0^1 D_{k-1} |(z - z_0)t|^k |z - z_0| \, dt$$

$$= D_{k-1} |z - z_0|^{k+1} \int_0^1 t^k \, dt$$

$$= \frac{D_{k-1}}{k+1} |z - z_0|^{k+1}.$$

I am a little suspecious about $|dw|$

in particular

is $|dw| = |z - z_0| dt$

So we could set $D_k = D_{k-1}/(k + 1)$. By induction, there exists D_k for all k. □

note $\frac{|w - z_0|^k}{|z - z_0|^k}$ is analytic if $z \ne z_0$

on $\overline{D}_{z_0}(r)$ compact set

so maximum exists. when $z = z_0$, all statement is trivially true.

C. **Proposition.** For for large k, we can estimate D_k by $1/(R^k(R-r))$.

Proof. Let $0 < r < R$ and $|z - z_0| \le r$. Pick any t with $r < t < R$. From the proof of the ratio test, we know that for some sufficiently large index k,

$$|a_n| < \left(\frac{1}{t}\right)^n \quad \text{for } n > k.$$

Then

$$\left| f(z) - \sum_{n=0}^{k} a_n (z - z_0)^n \right| = \left| \sum_{n=k+1}^{\infty} a_n (z - z_0)^n \right| \le \sum_{n=k+1}^{\infty} |a_n (z - z_0)^n| <$$

$$\sum_{n=k+1}^{\infty} \left(\frac{|z - z_0|}{t} \right)^n = |z - z_0|^{k+1} \frac{1}{t^{k+1}} \sum_{n=0}^{\infty} \left(\frac{|z - z_0|}{t} \right)^n \le |z - z_0|^{k+1} \frac{1}{t^{k+1}} \sum_{n=0}^{\infty} \left(\frac{r}{t} \right)^n .$$

Since this is true for any $t < R$,

$$\left| f(z) - \sum_{n=0}^{k} a_n (z - z_0)^n \right| \le |z - z_0|^{k+1} \frac{1}{R^{k+1}} \sum_{n=0}^{\infty} \left(\frac{r}{R} \right)^n ,$$

so let

$$D_k = \frac{1}{R^{k+1}} \sum_{n=0}^{\infty} \left(\frac{r}{R} \right)^n = \frac{1}{R^{k+1}} \cdot \frac{1}{1 - (r/R)} = \frac{1}{R^k (R - r)}.$$

This gives us an error bound of

$$\left| f(z) - \sum_{n=0}^{k} a_n (z - z_0)^n \right| \le D_k |z - z_0|^{k+1} \le \frac{r}{R - r} \left(\frac{r}{R} \right)^k .$$

□