Problem 2.5.7. Calculate $\int_\gamma 1/z\,\mathrm{d}z$ if γ is any path in $\mathbb{C}$ joining $-i$ to i which does not cross the half line $(-\infty, 0]$ on the real axis.

Solution. Let $\log(z)$ be the complex logarithm on the principal branch. By assumption, γ is a path in $\mathbb{C}\setminus(-\infty,0]$. Since $\log(z)$ is analytic with derivative $1/z$ on $\mathbb{C}\setminus(-\infty,0]$, Theorem 2.5.6 implies

$$\int_\gamma 1/z\,\mathrm{d}z = \log(i) - \log(-i) = i\pi/2 - (-i\pi/2) = i\pi.$$

Here we are assuming γ starts at $-i$ and ends at i. If it were the other way around, the integral would instead evaluate to $-i\pi$.

□

Problem 2.5.8. Show that

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = 0$$

if γ is any closed path contained in the complement of the set of non-positive real numbers. Compare this with Example 2.4.1.

Solution. Define $\log(z)$ as in Problem 2.5.7 so that it is analytic with derivative $1/z$ on $\mathbb{C}\setminus(-\infty,0]$. Since γ is a closed path in $\mathbb{C}\setminus(-\infty,0]$, Theorem 2.5.6 implies $\int_\gamma 1/z\,\mathrm{d}z = 0$. This result differs from Example 2.4.1 because no circle centered at 0 has a neighborhood on which $1/z$ has an antiderivative.

Problem 2.5.9. If $\sqrt{z}$ is defined by $\sqrt{z} = e^{(\log z)/2}$ for the branch of the log function defined by the condition $-\pi/2 \le \arg(z) \le 3\pi/2$, find an antiderivative for $\sqrt{z}$ and then find $\int_\gamma \sqrt{z}\,\mathrm{d}z$, where γ is any path from -1 to 1 which lies in the upper half-plane.

Solution. Let U be the open subset equal to $\mathbb{C}$ with the nonpositive imaginary axis removed. Since $\log(z)$ is analytic on U, the function $f: U \to \mathbb{C}$ give by $f(z) = \frac{2}{3}e^{3(\log z)/2}$ is analytic with derivative

$$f'(z) = \frac{2}{3}e^{3(\log z)/2}\cdot\frac{3}{2z} = \frac{e^{3(\log z)/2}}{e^{\log z}} = e^{(\log z)/2} = \sqrt{z},$$

as needed. The image of γ is in U, so Theorem 2.5.6 tells us that

$$\int_\gamma \sqrt{z}\,\mathrm{d}z = f(1) - f(-1) = \frac{2}{3}e^{3\cdot 0/2} - \frac{2}{3}e^{3\pi/2} = \frac{2}{3} + \frac{2}{3}i.$$

Problem 2.5.14. Show that Theorem 2.5.9 can be strengthened to conclude that the integral f around any triangle in U is 0 if f is continuous on U and analytic on $U \setminus I$, where I is an interval contained in U.

Lemma 1. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers converging to $z \in \mathbb{C}$. Choose $a \in \mathbb{C}$, and for each $n \in \mathbb{N}$, define $\gamma_n : [0,1] \to \mathbb{C}$ by $\gamma(t) = a + z_n t$. Finally, suppose $f : K \to \mathbb{C}$ is a continuous function on a compact set K. Then, the sequence $(z_n f \circ \gamma_n)_{n\in\mathbb{N}}$ converges uniformly to $zf \circ \gamma$.

Proof. Let $\epsilon > 0$. Since K is compact, f is uniformly continuous so there exists a $\delta > 0$ such that for all $z, w \in K$, $|z - w| < \delta$ implies $|f(z) - f(w)| < \epsilon/(2(|z|+1))$. Moreover, $f(K)$ is bounded so there exists a real number $M > 0$ such that $|f(z)| \leq M$ for all $z \in K$. Finally, choose $N \in \mathbb{N}$ such that $n \geq N$ implies $|z_n - z| < \min(\delta, \epsilon/(2M))$. Then, for all $t \in [0,1]$ and $n \geq N$,

$$|\gamma_n(t) - \gamma(t)| = |z_n - z||t| \leq |z_n - z| < \delta$$

and thus

$$\begin{aligned}
|z_n f \circ \gamma_n(t) - zf \circ \gamma(t)| &= |(z_n - z) f \circ \gamma_n(t) + z(f \circ \gamma_n(t) - f \circ \gamma(t))| \\
&\leq |z_n - z||f \circ \gamma_n(t)| + |z||f \circ \gamma_n(t) - f \circ \gamma(t)| \\
&< \frac{\epsilon}{2M} \cdot M + |z| \cdot \frac{\epsilon}{2(|z|+1)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

as desired. □

Solution. Suppose f is continuous on an open set U and analytic on $U \setminus I$ for some interval $I \subseteq U$. Let Δ be a triangle in U. If I and Δ are disjoint, then $U \setminus I$ is an open set containing Δ on which f is analytic and so $\int_{\partial\Delta} f(z)\,dz = 0$ by Theorem 2.5.8.

Now suppose $I \cap \Delta \neq \emptyset$, and let J be the interval $I \cap \Delta$. First, consider the case where J is one of the edges of Δ. Let a, b, and c denote the vertices of Δ (written in counterclockwise order with respect to some point in the interior of Δ) and assume without loss of generality that J is the edge joining a and b. Let $\alpha, \beta, \gamma : [0,1] \to \Delta$ be the straight line paths from a to b, b to c, and c to a respectively. For each $n \in \mathbb{N}$, let $p_n = c/n + (1 - 1/n)b$. Observe that $p_n \to b$ as $n \to \infty$. Then

$$\begin{aligned}
\int_{\partial\Delta} f(z)\,dz &= \int_\alpha f(z)\,dz + \int_\beta f(z)\,dz + \int_\gamma f(z)\,dz \\
&= \int_0^1 f(\alpha(t))\alpha'(t)\,dt + \int_0^1 f(\beta(t))\beta'(t)\,dt + \int_\gamma f(z)\,dz \\
&= \int_0^1 f(a + (b-a)t)(b-a)\,dt + \int_0^1 f(b + (c-b)t)(c-b)\,dt + \int_\gamma f(z)\,dz \\
&= \int_0^1 \lim_{n\to\infty} f(a + (p_n - a)t)(p_n - a)\,dt + \int_0^1 \lim_{n\to\infty} f(p_n + (c - p_n)t)(c - p_n)\,dt + \int_\gamma f(z)\,dz
\end{aligned}$$

(note that the last equality relies on the continuity of f in U). Since f is continuous on the compact set Δ, the lemma implies that the first two integrands converge uniformly. Thus we can interchange the integral and the limit, obtaining

$$\begin{aligned}
\int_{\partial\Delta} f(z)\,dz &= \lim_{n\to\infty} \int_0^1 f(a + (p_n - a)t)(p_n - a)\,dt + \lim_{n\to\infty} \int_0^1 f(p_n + (c - p_n)t)(c - p_n)\,dt + \int_\gamma f(z)\,dz \\
&= \lim_{n\to\infty} \left[\int_0^1 f(a + (p_n - a)t)(p_n - a)\,dt + \int_0^1 f(p_n + (c - p_n)t)(c - p_n)\,dt + \int_\gamma f(z)\,dz\right]
\end{aligned}$$

In order of appearance, the above integrals are the contour integrals of $f(z)$ along the straight lines from a to p_n, p_n to c, and c to a. Hence, when $n > 1$, their sum is equal to the contour integral of $f(z)$ along the boundary of the triangle with vertices a, p_n, and c. This triangle is contained in Δ but only intersects J at one point, namely a, so the integral along the boundary of this triangle is zero by Theorem 2.5.9. Thus

$$\int_{\partial\Delta} f(z)\,\mathrm{d}z = \lim_{n\to\infty} 0 = 0,$$

as needed.

On the other hand, if J is not an edge of Δ, then we can subdivide Δ into smaller triangles in such a way that any given triangle in this subdivision either has J as an edge, intersects J at a vertex, or does not intersect J at all. The integral along the boundary of any triangle having J as an edge is zero by our previous argument, and the integral along any other triangle is 0 by Theorem 2.5.9. The integral along $\partial\Delta$ is the sum of all of these integrals, so $\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0$. This completes the proof.

Problem 2.6.2. Calculate $\int_\gamma (z^2-4)^{-1}\,\mathrm{d}z$ if γ is the unit circle traversed once in the positive direction.

Solution. Let $U = D_{3/2}(0)$. Since U is an open set containing the unit circle but not containing 2 or -2, $(z^2-2)^{-1}$ is analytic on U and γ is a closed path in U. Then since U is convex, it follows from Theorem 2.6.2 that $\int_\gamma (z^2-4)^{-1}\,\mathrm{d}z = 0$.

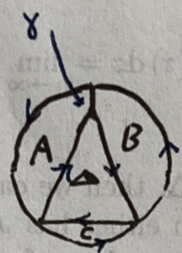
Problem 2.6.6. Show that the principal branch of the log function can be described by the formula $\int_1^z 1/w\,\mathrm{d}w$ for $z \notin (-\infty, 0]$.

Solution. Let $\log(w)$ be the complex logarithm on the principal branch with cut line $(-\infty, 0]$. For all $z \in \mathbb{C}\setminus(-\infty, 0]$, observe that the straight line path between 1 and z is contained in $\mathbb{C}\setminus(-\infty, 0]$. Since $\log(w)$ is analytic with derivative $1/w$ on $\mathbb{C}\setminus(-\infty, 0]$, Theorem 2.5.6 tells us that

$$\int_1^z 1/w\,\mathrm{d}w = \log(z) - \log(1) = \log(z).$$

Problem 2.6.8. Without doing any calculating, show that the integral of $1/z$ around the boundary of the triangle with vertices i, $1-i$, $-1-i$ is $2\pi i$.

Solution. Let Δ denote the triangle described in the problem statement. Let D denote the closed disk of radius $\sqrt{2}$ centered at 0, and subdivide D into regions according to the following graphic,



where the boundary of triangle is clockwisely oriented and the boundary of $D_{\sqrt{2}}(0)$ is counterclockwisely oriented.

By Example 2.4.1,

$$2\pi i = \int_{\partial D} 1/z \, dz$$

$$= \int_{\partial \Delta} 1/z \, dz + \int_{\partial A} 1/z \, dz + \int_{\partial B} 1/z \, dz + \int_{\partial C} 1/z \, dz.$$

For this equality to hold, not only you need $\int_{\partial A} f = \int_{\partial B} f = \int_{\partial C} f = 0$

Since $1/z$ is analytic an a convex neighborhood of A (take $\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 2\operatorname{Re}(z)\}$ for example), $\int_{\partial A} 1/z \, dz = 0$. Since the same can be said about B and C, we see that

$$\int_{\partial \Delta} 1/z \, dz = 2\pi i.$$

but also the orientation i.e. because we computed $\int_\gamma f$ and $\int_{-\gamma} f$ when we compute $\int_{\partial A} f$ and $\int_{\partial B} f$

Problem 2.6.12. Use Cauchy's Integral Formula to calculate $\int_{|z|=1} \frac{e^z}{z} \, dz$.

Solution. Let γ be any path tracing around the unit circle once in a counterclockwise direction. Then γ is a reparameterization of the path $t \mapsto e^{it}$ on $[0, 2\pi]$, so Example 2.4.1 shows that $\operatorname{Ind}_\gamma(0) = \frac{1}{2\pi i} \int_\gamma 1/z \, dz = 1$. Since e^z is analytic on the convex open set $\mathbb{C}$, Theorem 2.6.7 along with our previous calculation shows that

$$\int_\gamma \frac{e^z}{z} \, dz = 2\pi i e^0 = 2\pi i.$$

Problem 2.6.13. Use Cauchy's Formula to show that

$$\int_{|z-1|=1} \frac{1}{z^2-1}\,\mathrm{d}z = \pi i, \qquad \int_{|z+1|=1} \frac{1}{z^2-1}\,\mathrm{d}z = -\pi i.$$

Solution. Let $\gamma_1, \gamma_2 : I \to \mathbb{C}$ be closed paths traversing the unit circles centered at 1 and -1, respectively, in a counterclockwise direction. Note that $\mathrm{Ind}_{\gamma_1}(1) = \mathrm{Ind}_{\gamma_2}(-1) = 1$. Consider the functions $f_1(z) = 1/(z+1)$ and $f_2(z) = 1/(z-1)$. Since $f_1(z)$ and $f_2(z)$ are analytic on the convex open sets $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > -1\}$ containing $\gamma_1(I)$ and $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 1\}$ containing $\gamma_2(I)$, respectively, Theorem 2.6.7 implies

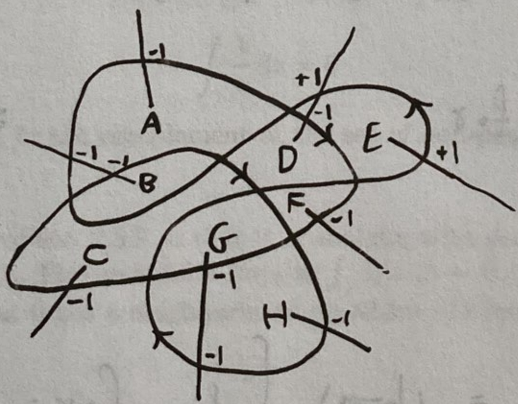
$$\int_{\gamma_1} \frac{1}{z^2-1}\,\mathrm{d}z = \int_{\gamma_1} \frac{f_1(z)}{z-1}\,\mathrm{d}z = 2\pi i f_1(1) = \pi i$$

and

$$\int_{\gamma_1} \frac{1}{z^2-1}\,\mathrm{d}z = \int_{\gamma_1} \frac{f_2(z)}{z+1}\,\mathrm{d}z = 2\pi i f_2(-1) = -\pi i.$$

Problem 2.7.9. Determine the value of $\mathrm{Ind}_\gamma(z)$ in each of the components of $\mathbb{C} \setminus \gamma(I)$ if γ is the curve of Figure 2.7.3.

Solution.



By inspection, the value of $\mathrm{Ind}_\gamma(z)$ in the components A, B, C, D, E, F, G, and H are -1, -2, -1, 0, 1, -1, -2, and -1, respectively.