Section 2.3

Problem 10

Let $\gamma(t) = 3e^{it}$ for $t \in [0, 2\pi]$.

(a)

$$\begin{aligned}
\int_\gamma z^{-1} dz &= \int_0^{2\pi} \frac{1}{\gamma(t)}\gamma'(t)dt \\
&= \int_0^{2\pi} \frac{1}{3}e^{-t} \cdot 3ie^{it} dt \\
&= \int_0^{2\pi} idt \\
&= 2\pi i
\end{aligned}$$

(b)

$$\begin{aligned}
\int_\gamma \bar{z} dt &= \int_0^{2\pi} \overline{\gamma(t)}\gamma'(t)dt \\
&= \int_0^{2\pi} 3e^{-it} \cdot 3ie^{it} dt \\
&= \int_0^{2\pi} 9idt \\
&= 18\pi i
\end{aligned}$$

Problem 13

No, it is not generally true that $\mathrm{Re}(\int_\gamma f(z)dz) = \int_\gamma \mathrm{Re}(f(z))dz$. For example, let $f(z) = 1$ and $\gamma(t) = it$ for $t \in [0, 1]$. Then

$$\int_\gamma \mathrm{Re}(f(z)dz) = \int_0^1 \mathrm{Re}(1)\gamma'(t)dt = \int_0^1 idt = i$$

This value is imaginary, so is clearly not equal to the value $\mathrm{Re}(\int_\gamma f(z)dz)$, which is a real number.

5

Problem 2.4.1. Compute $\int_\gamma z^2\,dz$ if γ is any path which traces once around the circle of radius 1, centered at 0, in the counterclockwise direction.

Solution. Assume the domain of γ is $[a,b]$ for some $a < b$. By assumption, we have $\gamma(a) = \gamma(b) = x$ for some point x on the unit circle. Since the polynomial $p(z) = z^3/3$ is analytic on $\mathbb{C}$ with derivative z^2, the fundamental theorem of calculus for contour integrals implies

↑
cts on $\mathbb{C}$

$$\int_\gamma z^2\,dz = p(x) - p(x) = 0.$$

Problem 4

(Section 2.4 Problem 2)

Compute $\int_\gamma 1/zdz$ if γ is any path which traces twice around the circle of radius one, centered at 0 in the counterclockwise direction.

Solution

Let γ be a path that traces around the unit circle twice with parameter interval $[0, 2\pi]$. Thus by the independence of parameterization, γ may be parameterized by the function $\gamma(t) = e^{2ti}, 0 \leq t \leq 2\pi$.

Therefore since $\frac{1}{z}$ is continuous and defined on $\gamma(t)$ $(\gamma(t) \neq 0 \forall t \in [0, 2\pi])$,

$$\begin{aligned}\int_\gamma 1/zdz &= \int_0^{2\pi} e^{-2ti} \cdot 2ie^{2ti}dt \\ &= \int_0^{2\pi} 2idt \\ &= 4\pi i\end{aligned}$$

Therefore, $\int_\gamma 1/zdz = 4\pi i$ if γ is any path which traces twice around the circle of radius one, centered at 0 in the counterclockwise direction.

8. **Proposition.** If γ is a path that traces the unit circle once, then

$$\left| \int_\gamma \frac{\cos z}{z}\, dz \right| \leq 2\pi e.$$

Because γ lies the unit circle ,

Proof. ~~Let~~ $|z| = 1$. By the triangle inequality and Theorem 1.3.7(c),

$$|\cos z| = \left| \frac{e^{iz} + e^{-iz}}{2} \right| \leq \frac{|e^{iz}|}{2} + \frac{|e^{-iz}|}{2} \leq \frac{e^{|iz|}}{2} + \frac{e^{|-iz|}}{2} = \frac{e^1}{2} + \frac{e^1}{2} = e,$$

and so $|(\cos z)/z| = |\cos z|/|z| \leq e$. Then by Theorem 2.4.9,

$$\left| \int_\gamma \frac{\cos z}{z}\, dz \right| \leq e \cdot \ell(\gamma) = 2\pi e,$$

where in the last step, we used that arc length of the unit circle γ is 2π.

Problem 10

Let $p(z)$ be an arbitrary polynomial in p and let $\gamma : [0, 2\cdot\pi] \to \mathbb{C}$ defined by $\gamma(t) = e^{i\cdot t}$ be a path, and note that $\gamma(0) = \gamma(2\cdot\pi)$. We will show that $\int\limits_{\gamma} p(z)\cdot dz = 0$. We can write $p(z) = \sum\limits_{i=0}^{n} (a_i\cdot z^i)$ for some $n \in \mathbb{N}$ where $a_i \in \mathbb{C}$ for $i \in$ 0, 1, ..., n $\cancel{[0, \pi]}$ and $a_n \neq 0$. Note that for all $i \in [0, n]$ and for all $z \in \mathbb{C}$, $\frac{a_i\cdot z^{i+1}}{i+1}$ is analytic and continuously differentiable and $\left(\frac{a_i\cdot z^{i+1}}{i+1}\right)' = a_i\cdot z^i$, and thus by the fundamental theorem of calculus for curves, $\int\limits_{\gamma} p(z)\cdot dz = \int\limits_{\gamma} \sum\limits_{i=0}^{n} (d_i\cdot z^i)\cdot dz = \sum\limits_{i=0}^{n} \left(\int\limits_{\gamma} a_i\cdot z^i\cdot dz \right) = \sum\limits_{i=0}^{n} \left(\frac{a_i\cdot\gamma(2\cdot\pi)^{i+1}}{i+1} - \frac{a_i\cdot\gamma(0)^{i+1}}{i+1} \right) = \sum\limits_{i=0}^{n} \left(\frac{a_i\cdot\gamma(0)^{i+1}}{i+1} - \frac{a_i\cdot\gamma(0)^{i+1}}{i+1} \right) = \sum\limits_{i=0}^{n} (0) = 0.$