2.1.5

(a) $\{z \in \mathbb{C} : 1 < |z| < 2\}$ is open.

(b) $\{z \in \mathbb{C} : \Im(z) = 0, 0 < \Re(z) < 1\}$ is neither open nor closed.

(c) $\{z \in \mathbb{C} : -1 \leq \Re(z) \leq 1, -1 \leq \Im(z) \leq 1\}$ is closed.

2.1.6

Let $E = \{z \in \mathbb{C} : 1 \leq |z| < 2\}$. Then:

- The interior is $E^\circ = \{z \in \mathbb{C} : 1 < |z| < 2\}$;
- The exterior is $\overline{E} = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$;
- The boundary is $\partial E = \{z \in \mathbb{C} : |z| = 1 \text{ or } |z| = 2\}$.

Problem 2.1.13. Use Theorem 2.1.13 to prove that if f and g are continuous functions with open domains U_f and U_g and if $g(U_g) \subset U_f$, then $f \circ g$ is continuous on U_g.

Solution. Suppose $W \subseteq \mathbb{C}$ is open. Since f is continuous, Theorem 2.1.13 shows that $f^{-1}(W)$ is open. Then, the continuity of g implies that $g^{-1}(f^{-1}(W)) = (f \circ g)^{-1}(W)$ is open, and it follows from Theorem 2.1.13 that $f \circ g$ is continuous.

2

5

Problem 2.1.14. Prove that if f is a continuous function defined on an open subset U of $\mathbb{C}$, then sets of the form $\{z \in U \mid |f(z)| < r\}$ and $\{z \in U \mid \text{Re}(f(z)) < r\}$ are open.

Solution. Suppose $U \subseteq \mathbb{C}$ is open and $f : U \to \mathbb{C}$ is a continuous function. Let $r \in \mathbb{R}$ and $V = \{z \in \mathbb{C} \mid \text{Re}(z) < r\}$. First, we prove that V is open by showing that for all $v \in V$, $D_{r-\text{Re}(v)}(v) \subseteq V$. Letting $w \in D_{r-\text{Re}(v)}(v)$ be arbitrary, we have

$$\begin{aligned} r &= r - \text{Re}(v) + \text{Re}(v) \\ &> |w - v| + \text{Re}(v) \\ &\geq \text{Re}(w - v) + \text{Re}(v) \\ &= \text{Re}(w - v + v) \\ &= \text{Re}(w), \end{aligned}$$

so $w \in V$. Hence, V is open. Since

$$f^{-1}(D_r(0)) = \{z \in U \mid f(z) \in D_r(0)\} = \{z \in U \mid |f(z)| < r\}$$

and

$$f^{-1}(V) = \{z \in U \mid f(z) \in V\} = \{z \in U \mid \text{Re}(f(z)) < r\},$$

the result follows from Theorem 2.1.13.

2.1.16

$\Rightarrow$: Suppose $f : U \to \mathbb{C}$ is continuous at a point $a \in U$. Let $\{z_n\} \subset U$ be a sequence converging to a. Let $\epsilon > 0$. By the continuity of f, $\lim_{z \to a} f(z) = f(a)$. By the definition of limit, there exists $\delta > 0$ such that $|f(z) - f(a)| < \epsilon$ whenever $z \in U$ and $|z - a| < \delta$. By the convergence of $\{z_n\}$, there exists an index N such that for all $n \geq N$, we have $|z_n - a| < \delta$. Therefore, for all $n \geq N$, we have $|f(z_n) - f(a)| < \epsilon$, so that $\{f(z_n)\}$ converges to $f(a)$.

$\Leftarrow$: Suppose that whenever $\{z_n\} \subset U$ is a sequence converging to a, $\{f(z_n)\}$ converges to $f(a)$. Suppose further for the sake of contradiction that f is not continuous at a. By the definition of continuous, $\lim_{z \to a} f(z) \neq f(a)$. By the definition of limit, there exists some $\epsilon > 0$ such that there exists no $\delta > 0$ for which $|f(z) - f(a)| < \epsilon$ whenever $|z - a| < \delta$. Thus for each natural number n, there is a point $z_n \in U$ such that $|z_n - a| < 1/n$ but $|f(z_n) - f(a)| \geq \epsilon$. This defines a sequence $\{z_n\} \subset U$ that converges to a. By our first assumption, $\{f(z_n)\}$ must converge to $f(a)$, which contradicts the assertion that $|f(z_n) - f(a)| \geq \epsilon$ for each natural number n. This contradiction implies our second assumption was false, and thus f is continuous at a.

Sec 2.2. Problem. 5, 11, 14.

5. Use induction and Theorem 2.2.6 to show that $(z^n)' = nz^{n-1}$ if n is a non-negative integer.

Proof: Let $f = z^n$ for $n \geq 0$.
We will prove by induction that $f'(z) = nz^{n-1}$
Basis case:
Let $n = 0$. then $f = z^0 = 1$. and $f'(z) = \lim_{\lambda \to 0} \frac{f(z+\lambda) - f(z)}{\lambda}$
$= \lim_{\lambda \to 0} \frac{1-1}{\lambda} = 0$

Let $n = 1$. then $f(z) = z$

$$f'(z) = \lim_{\lambda \to 0} \frac{f(z+\lambda) - f(z)}{\lambda} = \frac{z+\lambda-z}{\lambda} = 1 = 1 \cdot z^{1-1} = z^0$$

True for $n = 1$.
Induction Step: Assume the statement holds for any $k = n$.
we want to show this is true for $k+1 = n$.
Let $g = z^{k+1} = z^k \cdot z$
By theorem 2.2.6

$$\begin{aligned} g' &= (z^k)' z + z^k \cdot (z') \\ &= k z^{k-1} \cdot z + z^k \cdot 1 \\ &= k z^{k-1+1} + z^k \\ &= k z^k + z^k \\ &= z^k (k+1) \end{aligned}$$

Thus. the statement is true for $n = k+1$.
This completes the induction process.
Therefore. we've proved that
$(z^n)' = nz^{n-1}$ for all nonnegative integer n.

☑

$$= nz^n + z^n$$
$$= (n+1)z^{(n+1)-1}$$

Thus, the claim holds for $n+1$. We have shown that $(z^n)' = nz^{n-1}$ for all $n \in \mathbb{N}$ by induction. ∎

Problem 11

Proof. Let $f : \mathbb{C} \to \mathbb{R}$ be analytic. If we write $f(x+iy) = u(x,y) + iv(x,y)$ for some functions $u, v : \mathbb{R}^2 \to \mathbb{R}$, then we know that both u and v are differentiable and that they satisfy the Cauchy-Riemann equations. That is, $u_x = v_y$ and $u_y = -v_x$. Now since f is a real-valued function, we know that v must be identically zero. So, $v_x = v_y = 0$. Thus it follows that $u_x = u_y = 0$.

Since u is differentiable, it follows from the Mean Value Theorem that u is a constant function. That is, since $\nabla u(x,y) = \mathbf{0}$, the Mean Value Theorem gives $u(s,t) - u(x,y) = \mathbf{0} \cdot ((s,t) - (x,y))$ for any $(s,t), (x,y) \in \mathbb{R}^2$. So $u(s,t)$ and $u(x,y)$ must be equal and hence $u(x,y) = c$ for some $c \in \mathbb{R}$.

In conclusion, we have deduced that u is constant, and v is identically zero. Thus, $f(x+iy) = u(x,y) + iv(x,y) = c$ must be a constant function. ∎

Problem 2.2.14. Without assuming each branch of the log function is analytic, use the chain rule to give another proof that each such function has derivative $1/z$.

Solution. Let $\log : C \to \mathbb{C}$ denote the complex natural logarithm on some arbitrary branch C. Since $\log(z)$ is continuous, $\lim_{w\to z} \log(w) = \log(z)$ for all $z \in C$. Then, since the exponential function is analytic with nonzero derivative on $\mathbb{C}$, it follows that

$$\lim_{w\to z} \frac{e^{\log(w)} - e^{\log(z)}}{\log(w) - \log(z)} = \lim_{w\to z} \frac{w - z}{\log(w) - \log(z)}$$

exists and is nonzero for all $z \in C$. Thus $\lim_{w\to z} \frac{\log(w)-\log(z)}{w-z}$ also exists for all $z \in C$, that is, $\log(z)$ is analytic on C.

Now, note that $z = e^{\log(z)}$ for all $z \in C$, and since $\log(z)$ and e^z are both analytic, we can differentiate both sides of this equation to obtain

$$1 = (\log(z))' e^{\log(z)} = (\log(z))' z.$$

Since $0 \notin C$, we have $z \neq 0$ and it follows that $(\log(z))' = 1/z$ for all $z \in C$.