Math 427 homework 2

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Problem 1.3.3

Proof. Write $z = x + iy$ and then $e^z = e^x e^{iy} = e^x(\cos y + i \sin y) = 1/\sqrt{2} + i/\sqrt{2}$. Taking the norm of both sides, we get $e^x = 1$ and hence $x = 0$. Now equating the real and the imaginary part on both side, we get $\cos y = 1/\sqrt{2} = \sin y$ which implies $y = \pi/4$. Thus, $z = i\pi/4$. $\square$

Problem 1.3.11

Proof. The base case is obvious when $n = 0$ or $n = 1$. Now suppose $(z+w)^k = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} z^j w^{k-j}$ holds for $n = k$. We will show $(z+w)^{k+1} = \sum_{j=0}^{k+1} \frac{(k+1)!}{j!(k+1-j)!} z^j w^{k+1-j}$. Note that

$$\begin{aligned}
(z+w)^{k+1} = (z+w)^k(z+w) &= \Big(\sum_{j=0}^{k} \frac{k!}{j!(k-j)!} z^j w^{k-j}\Big)(z+w) \\
&= \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} z^{j+1} w^{k-j} + \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} z^j w^{k+1-j} \\
&= \sum_{i=1}^{k+1} \frac{k!}{(i-1)!(k+1-i)!} z^i w^{k+1-i} + \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} z^j w^{k+1-j} \\
&= w^{k+1} + \sum_{j=1}^{k} \Big(\frac{k!}{(j-1)!(k+1-j)!} + \frac{k!}{j!(k-j)!}\Big) z^j w^{k+1-j} + z^{k+1} \\
&= w^{k+1} + \sum_{j=1}^{k} \frac{(k+1)!}{j!(k+1-j)!} z^j w^{k+1-j} + z^{k+1} = \sum_{j=0}^{k+1} \frac{(k+1)!}{j!(k+1-j)!} z^j w^{k+1-j}.
\end{aligned}$$

By induction, we conclude the complex binomial formula. $\square$

Lemma 1. *Problem 1.3.10*

Proof. Consider the real power series $\sum a_n x^n$ with radius of convergence R and the complex power series $\sum a_n z^n$ with radius of convergence R'. It follows immediately $R' \leq R$. Now let $z \in D_0(R)$ and write $z = xe^{i\theta}$ for some $x \in (-R, R)$ and $0 \leq \theta < 2\pi$. Note that $\sum |a_n z^n| = \sum |a_n x^n|$ and hence converges absolutely. It follows $R \leq R'$. We conclude $R = R'$. □

Problem 1.3.15

Proof. Recall $\log(1+x) = \int_0^x \frac{1}{1+t} = \int_0^x \sum_{i=0}^{\infty}(-1)^i t^i = \sum_{i=0}^{\infty} \int_0^x (-1)^i t^i = \sum_{i=0}^{\infty}(-1)^i \frac{1}{i+1} x^{i+1}$ on $(-1, 1)$. Then by Lemma 1, the complex power series $\sum_{i=0}^{\infty}(-1)^i \frac{1}{i+1} z^{i+1}$ also has radius of convergence 1. So we define the complex function $\log(1+z)$ on $D_1(0)$ via $\sum_{i=0}^{\infty}(-1)^i \frac{1}{i+1} z^{i+1}$. □

Problem 1.4.3

Proof. Note $(e^{i\pi/8})^{16} = 1$. So for any $k \in \mathbb{Z}$, we write $(e^{i\pi/8})^k = e^{ik\pi/8} = e^{i(16p+q)\pi/8} = e^{iq\pi/8}$ where $p \in \mathbb{Z}$ and $q = 0, 1, \cdots, 15$. One may see geometrically that there are 16 distinct points evenly distributed on the unit circle spaced apart by an angle of $\pi/8$. □

Problem 1.4.9

Proof. Assume $z \neq 0$. Fix w to be a nth root of z. Let ξ be an arbitrary nth root of z. Since $\xi, w \neq 0$ and $\mathbb{C}$ is a field, write $\xi = w\alpha$ for some $0 \neq \alpha \in \mathbb{C}$. It suffices to show α is a root of unity. Note that $z = \xi^n = (w\alpha)^n = w^n \alpha^n$ which implies $\alpha^n = 1$. □

Problem 1.4.18

Proof. Write $z = e^{i\theta} = \cos\theta + i\sin\theta$ for some $0 \leq \theta < 2\pi$. It follows that $e^{-i\theta} - e^{i\theta} = -2i\sin\theta$. Then we may rewrite the identity as

$$\begin{aligned}
1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \\
&= \frac{e^{i(n+1)\theta/2}(e^{-i(n+1)\theta/2} - e^{i(n+1)\theta/2})}{e^{i\theta/2}(e^{-i\theta/2} - e^{i\theta/2})} \\
&= \frac{e^{i(n+1)\theta/2}}{e^{i\theta/2}} \frac{-2i\sin(n+1)\theta/2}{-2i\sin\theta/2} = e^{in\theta/2}\frac{\sin(n+1)\theta/2}{\sin\theta/2}.
\end{aligned}$$

Taking the real part of both sides, we get $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \cos n\theta/2 \frac{\sin(n+1)\theta/2}{\sin\theta/2}$. Recall $\cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$ for any A, B. Thus,

$$\begin{aligned}\frac{\cos n\theta/2 \sin(n+1)\theta/2}{\sin\theta/2} &= \frac{1}{2}\frac{(\sin(n\theta/2 + (n+1)\theta/2) - \sin(n\theta/2 - (n+1)\theta/2))}{\sin\theta/2} \\ &= \frac{1}{2}\frac{\sin(n+1/2)\theta + \sin\theta/2}{\sin\theta/2} = \frac{1}{2} + \frac{\sin(n+1/2)\theta}{2\sin\theta/2}.\end{aligned}$$

□