Math 427 homework 1

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Problem 1

proof of part a. Let $z \in \mathbb{C} - \{0\}$ be nonzero complex number. Recall its multiplicative inverse $\frac{1}{z}$ is the (unique) complex number w such that $wz = 1$. Thus, $1/\overline{z}$ is the (unique) complex number w such that $w\overline{z} = 1$. Note that $\overline{\frac{1}{z}} \cdot \overline{z} = \overline{\frac{1}{z} \cdot z} = \overline{1} = 1$. So $1/\overline{z}$ is the (unique) multiplicative inverse of $\overline{z}$. □

proof of part b. Since $z \cdot \frac{1}{z} = 1$, taking norm on both sides of the equality, we get $|z \cdot \frac{1}{z}| = |1|$ which implies $|z| \cdot |\frac{1}{z}| = 1$. It follows $|\frac{1}{z}| = \frac{1}{|z|}$. □

proof of part c. Note that $|z/\overline{z}| = |z| \cdot |\frac{1}{\overline{z}}| = |\overline{z}| \cdot |\frac{1}{\overline{z}}| = 1$ by $|z| = |\overline{z}|$ and part b (replacing z with $\overline{z}$). □

Problem 2

Proof. Recall $|z|^2 = z\overline{z}$ for any $z \in \mathbb{C}$. So $|z+w|^2 = (z+w)(\overline{z}+\overline{w})$ and similarly $|z-w|^2 = (z-w)(\overline{z}-\overline{w})$. Expand the product and we see $|z+w|^2 + |z-w|^2 = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} + z\overline{z} - z\overline{w} - w\overline{z} + w\overline{w} = 2(|z|^2 + |w|^2)$. □

Problem 3

Proof. By triangle inequality, $|w| = |w - z + z| \leq |w - z| + |z| = |z - w| + |z|$. This implies $|z - w| \geq |w| - |z|$. Similarly, $|z| = |z - w + w| \leq |z - w| + |w|$ and hence $|z - w| \geq |z| - |w|$. It follows $|z - w| \geq ||z| - |w||$. □

Problem 4

Proof. We prove by contradiction. Suppose $\sum z_n$ converges to some complex number $w \in \mathbb{C}$. This means the partial sum sequence $\{s_k\}$ converges to some w in $\mathbb{C}$, where $s_k = \sum_{i=1}^{k} z_i$. It follows that $\lim_{k\to\infty} z_k = \lim_{k\to\infty} (s_k - s_{k-1}) = \lim_{k\to\infty} s_k - \lim_{k\to\infty} s_{k-1} = w - w = 0$. □

Problem 5

Proof. Note that $\lim_{n\to\infty} \frac{|\frac{z^{n+1}}{2^{n+1}-1}|}{|\frac{z^n}{2^n-1}|} = \lim_{n\to\infty} |z| \cdot \lim_{n\to\infty} \frac{2^n - 1}{2^{n+1} - 1} = \frac{|z|}{2}$ which is less than 1 if $|z| < 2$. By ratio test, we know the series converges absolutely on $|z| < 2$ and diverges on $|z| > 2$.

Now suppose $|z| = 2$. Note for $n \geq 1$, we get $\lim_{n\to\infty} |\frac{z^n}{2^n - 1}| = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = 1 \neq 0$. This implies $\lim_{n\to\infty} \frac{z^n}{2^n - 1} \neq 0$ since it has a nonzero norm. By question 4, the series diverges on $|z| = 2$. □

Problem 6

Proof. We first note that $n^2 + z^2 = (z - in)(z + in)$. So the series is not defined on $i\mathbb{Z}$.

Next, we show the series converges on $\mathbb{C} - i\mathbb{Z}$. Fix $z \in \mathbb{C} - i\mathbb{Z}$ and note that there exists some $N \in \mathbb{N}$ such that $|z| < N/\sqrt{2}$. So when $N < n$ (and hence $|z| < n$), it follows from question 3 that

$$|n^2 + z^2| \geq ||n^2| - |z^2|| = n^2 - |z|^2 \geq n^2 - N^2/2 \geq n^2 - n^2/2 = n^2/2$$

which implies $|\frac{1}{n^2+z^2}| \leq \frac{2}{n^2}$. Thus, the tail $\sum_{n=N+1}^{\infty} \frac{1}{n^2+z^2}$ converges absolutely by comparison test. It follows $\sum_{n=1}^{\infty} \frac{1}{n^2+z^2}$ converges (absolutely). □