2 Analytic Functions

2.1 Polynomials

This course is about complex-valued functions of a complex variable. We could think of such functions in terms of real variables as maps from $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$f(x,y) = (u(x,y), v(x,y)),$$

and think of the graph of f as a subset of $\mathbb{R}^4$. But the subject becomes more tractable if we use a single letter z to denote the independent variable and write $f(z)$ for the value at z, where $z = x + iy$ and $f(z) = u(z) + iv(z)$. For example,

$$f(z) = z^n$$

is much simpler to write (and understand) than its real equivalent. Here z^n means the product of n copies of z.

The simplest functions are the **polynomials** in z:

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n, \tag{2.1}$$

where $a_0, \ldots, a_n$ are complex numbers. If $a_n \neq 0$, then we say that n is the **degree** of p. Note that $\bar{z}$ is not a (complex) polynomial, and neither is $\text{Re}z$ or $\text{Im}z$.

Let's take a closer look at **linear** or degree 1 polynomials. For example, if b is a (fixed) complex number, then

$$g(z) = z + b,$$

translates, or shifts, the plane. If a is a (fixed) complex number then

$$h(z) = az$$

can be viewed as a dilation and rotation. To see this, recall that by Chapter 1 and Exercise 1.2, $|az| = |a||z|$ and $\arg(az) = \arg(a) + \arg(z)$ (up to a multiple of 2π). So, h dilates z by a factor of $|a|$ and rotates the point z by the angle $\arg a$. A linear function

$$f(z) = az + b$$

can then be viewed as a dilation and rotation followed by a translation. Equivalently, writing $f(z) = a(z + b/a)$ we can view f as a translation followed by a rotation and dilation.

Another instructive example is the function $p(z) = z^n$. By Chapter 1 again,

$$|p(z)| = |z|^n \quad \text{and} \quad \arg p(z) = n \arg z \mod 2\pi.$$

Each pie slice

$$S_k = \left\{ z : \left| \arg z - \frac{2\pi k}{n} \right| < \frac{\pi}{n} \right\} \cap \{z : |z| < r\},$$

$k = 0, \ldots, n-1$ is mapped to a slit disk

$$\{z : |z| < r^n\} \setminus (-r^n, 0).$$

Angles between straight-line segments issuing from the origin are multiplied by n, and, for small r, the size of the image disk is much smaller than the "radius" of the pie slice. See Figure 2.1.

The function $k(z) = b(z - z_0)^n$ can be viewed as a translation by $-z_0$, followed by the power function, and then a rotation and dilation. To put it another way, k translates a neighborhood of z_0 to the origin, then acts like the power function z^n, followed by a dilation and rotation by b.

To understand the local behavior of a polynomial (2.1) near a point z_0, write $z = (z - z_0) + z_0$ and expand (2.1) by multiplying out and collecting terms to obtain

$$p(z) = p(z_0) + b_1(z - z_0) + b_2(z - z_0)^2 + \ldots + b_n(z - z_0)^n. \tag{2.2}$$

Another way to see this is to note that $p(z) - a_n(z - z_0)^n$ is a polynomial of degree at most $n - 1$, so (2.2) follows by induction on the degree. If $b_1 \neq 0$ then $p(z)$ behaves like the linear function $p(z_0) + b_1(z - z_0)$ for z near z_0. If $b_1 = 0$ then, near z_0, $p(z)$ is closely approximated by $p(z_0) + b_k(z - z_0)^k$, where b_k is the first non-zero coefficient in the expansion (2.2). Indeed, for small $\zeta = z - z_0$,

$$|p(z_0 + \zeta) - [p(z_0) + b_k\zeta^k]| \leq C|\zeta|^{k+1},$$

for some constant C, by (2.2). Figure 2.2 is sometimes called "walking the dog," where the walking path has radius $r = |b_k||\zeta|^k$ and the leash has length $s = C|\zeta|^{k+1}$. As ζ traces a circle centered at 0 of radius ε, the function $p(z_0) + b_k\zeta^k$ winds k times around the circle centered at $p(z_0)$ with radius r. For small ε, s is much smaller than r so the function $p(z_0 + \zeta)$ also then traces a path which winds k times around $p(z_0)$.

So, for z near z_0, $p(z)$ behaves like a translation by $-z_0$, followed by a power function, a rotation and dilation, and finally a translation by $p(z_0)$.

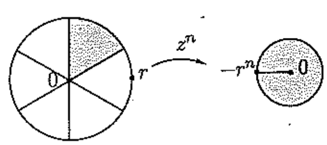

Figure 2.1 The power map.

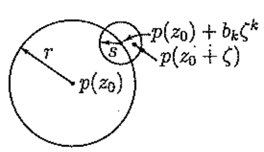

Figure 2.2 $p(z_0 + \zeta)$ lies in a small disk of radius $s = C|\zeta|^{k+1} < r = |b_k||\zeta|^k$.

2.2 Fundamental

The local behavior of a polynom result you will have seen in son $p(a) = 0$ then a is called a **zero**

Theorem 2.1 (fundamental th *zero.*

This remarkable result says t the solution to the equation z^2 -

Proof Suppose $p(z) = a_n z^n$ · has no zeros and for which a_n value on $\mathbb{C}$. Write

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Because $a_n \neq 0$, $1/z^k \to 0$ a $|z| \to \infty$. So, if $M = \inf_{\mathbb{C}} |p(z$ for all j. Because $\{z : |z| \leq R$ that $|p(z_0)| = M$. Moreover, argument in Section 2.1 (see F function $p(z_0 + \zeta)$ traces a path radius M because $|p(z_0)| = M$ r, s are defined in Section 2.1. $|p(z_0) + b_k\zeta^k| = M - r$. Then

$|p(z_0 + \zeta)| \leq |p(z_0$

This contradiction proves that

Corollary 2.2 *If p is a po* $z_1, \ldots, z_n$ *and a complex cons*

Corollary 2.2 does not tell zeros, counting multiplicity.

Proof The proof is by induc

$z^k - b^k =$

So if $p(z) = \sum_{k=0}^{n} a_k z^k$ and p

2.2 Fundamental Theorem of Algebra and Partial Fractions

The local behavior of a polynomial described in Section 2.1 can be used to prove an important result you will have seen in some form or another since high school. If p is a polynomial and $p(a) = 0$ then a is called a **zero** of p.

Theorem 2.1 (fundamental theorem of algebra) *Every non-constant polynomial has a zero.*

This remarkable result says that if we extend the real numbers to the complex numbers via the solution to the equation $z^2 + 1 = 0$ then every polynomial equation has a solution.

Proof Suppose $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$, $n \geq 1$, is a polynomial which has no zeros and for which $a_n \neq 0$. We first prove that $|p(z)|$ must have a non-zero minimum value on $\mathbb{C}$. Write

$$p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right).$$

Because $a_n \neq 0$, $1/z^k \to 0$ and $|z^n| \to \infty$ as $|z| \to \infty$, we conclude that $|p(z)| \to \infty$ as $|z| \to \infty$. So, if $M = \inf_{\mathbb{C}} |p(z)|$ and $|p(w_j)| \to M$, then there is an $R < \infty$ so that $|w_j| \leq R$, for all j. Because $\{z : |z| \leq R\}$ is compact and because $|p|$ is continuous, there exists z_0 so that $|p(z_0)| = M$. Moreover, $M \neq 0$ since p has no zeros. Now, by the "walking the dog" argument in Section 2.1 (see Figure 2.2), for small $\varepsilon > 0$, as ζ traces a circle of radius ε, the function $p(z_0 + \zeta)$ traces a path about $p(z_0)$ which must intersect the open disk centered at 0 of radius M because $|p(z_0)| = M$. More explicitly, choose $\varepsilon > 0$ so that $0 < s < r < M$, where r, s are defined in Section 2.1. Because $b_k \zeta^k$ traces a circle of radius r, we can find ζ so that $|p(z_0) + b_k \zeta^k| = M - r$. Then

$$|p(z_0 + \zeta)| \leq |p(z_0 + \zeta) - (p(z_0) + b_k \zeta^k)| + M - r \leq s + M - r < M.$$

This contradiction proves that no such polynomial exists and Theorem 2.1 follows. □

Corollary 2.2 *If p is a polynomial of degree $n \geq 1$, then there are complex numbers $z_1, \ldots, z_n$ and a complex constant c so that*

$$p(z) = c \prod_{k=1}^{n} (z - z_k).$$

Corollary 2.2 does not tell us how to find the zeros, but it does say that there are exactly n zeros, counting multiplicity.

Proof The proof is by induction. First note that

$$z^k - b^k = (z - b)(b^{k-1} + z b^{k-2} + \ldots + z^{k-2} b + z^{k-1}).$$

So if $p(z) = \sum_{k=0}^{n} a_k z^k$ and $p(b) = 0$, then

$$q(z) \equiv \frac{p(z)}{z-b} = \frac{p(z)-p(b)}{z-b} = \sum_{k=1}^{n} a_k \left(\sum_{j=0}^{k-1} b^{k-1-j} z^j \right). \tag{2.3}$$

The coefficient of z^{n-1} in (2.3) is a_n so q is a polynomial of degree $n-1$. Repeating this argument n times proves the corollary. □

For example, the polynomial $z^n - 1$ has n zeros. If $z^n = 1$ then $1 = |z^n| = |z|^n$ so that $|z| = 1$. Write $z = \cos t + i \sin t = e^{it}$ (see Exercise 2.6). Then $z^n = e^{int} = 1$ so that $nt = 2\pi k$ for some integer k, and thus $t = 2\pi k/n$. The n distinct zeros of $z^n - 1$ are then $e^{i2\pi k/n}$, $k = 0, 1, \ldots, n-1$, which are equally spaced around the unit circle.

A **rational function** r is the ratio of two polynomials. By the fundamental theorem of algebra, we can write r in the form

$$r(z) = \frac{p(z)}{\prod_{j=1}^{N}(z - z_j)^{n_j}}.$$

The next corollary, also probably familiar, allows us to write a rational function in a form that is easier to analyze. The form is also of practical importance because it allows us to solve certain differential equations that arise in engineering problems using the Laplace transform and its inverse.

Corollary 2.3 (partial fraction expansion) *If p is a polynomial then there is a polynomial q and constants $c_{k,j}$ so that*

$$\frac{p(z)}{\prod_{j=1}^{N}(z - z_j)^{n_j}} = q(z) + \sum_{j=1}^{N} \sum_{k=1}^{n_j} \frac{c_{k,j}}{(z - z_j)^k}. \tag{2.4}$$

Proof There are two initial cases to consider: If p is a polynomial then

$$\frac{p(z)}{z-a} = q(z) + \frac{p(a)}{z-a}, \tag{2.5}$$

where $q(z) = (p(z) - p(a))/(z - a)$ is a polynomial, as in (2.3). Secondly, if $a \neq b$, we can write

$$\frac{1}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b}, \tag{2.6}$$

for some constants A and B. For if this equation is true, then we can multiply each term on the right by $z - a$ and let $z \to a$ to obtain A on the right. The same process on the left yields $1/(a-b)$, and hence $A = 1/(a-b)$. Similarly $B = 1/(b-a)$. Now substitute these values for A and B into (2.6) and check that equality holds. The full corollary now follows by induction: suppose the corollary is true if the degree of the denominator is at most d. If we have an equation of the form (2.4) of degree d then we can divide each term in the equation by $z - a$. After division, the right-hand side consists of lower degree terms to which the induction hypothesis applies, with one exception: when the denominator of the left-hand side of (2.4) is $(z-b)^d$. If $a = b$, then, after division by $z - a$, each term will be of the correct form. If $a \neq b$, then we could have applied the inductive assumption to the decomposition of

$$\frac{p(z)}{(z-b)^{d-1}(z-a)}$$

and then divided this result by $z-b$ instead of $z-a$. □

The above proof also suggests an algorithm for computing the coefficients $\{c_{k,j}\}$. First apply (2.5) with $a = z_1$. Multiply each term of the result by $1/(z-b)$, where b is one of the zeros of the denominator in (2.4), and apply either (2.5) or (2.6) to each of the resulting terms on the right-hand side. Repeat this process, increasing the degree of the denominator by 1 until you have the desired expansion. At each stage, the terms to be expanded are of the form $1/[(z-a)^k(z-b)]$. These can be expanded by starting with (2.6), dividing the result by $(z-a)$, then using (2.6) again, repeating until you have reached the power k in the denominator.

The algorithm can be speeded up because we know the form of the solution. If powers in the denominator n_j are all equal to one, and if the numerator has smaller degree than the denominator, then the form is

$$\frac{p(z)}{\prod_{j=1}^{N}(z-z_j)} = \sum_{j=1}^{N} \frac{c_j}{z-z_j}. \tag{2.7}$$

If we multiply each term of the right-hand side by $z-z_1$ then let $z \to z_1$, we obtain c_1. If we multiply the left-hand side by the same factor, it cancels one of the terms in the denominator. Letting $z \to z_1$ we obtain the value of the remaining part of the left-hand side at z_1. This quickly gives c_1 and can be repeated for $c_2, \ldots, c_N$. This method is sometimes called the "cover-up method" because it can be done with less writing by observing that c_j is the value of the left-hand side at z_j when you cover $z - z_j$ with your hand. If the denominator has terms with degree bigger than one, first use a denominator with all terms of degree one as above. Then, as in the proof of Corollary 2.3, multiply everything by $1/(z-z_k)$ and simplify all terms on the right, repeating as often as needed. If the degree of the numerator of any term is not less than the degree of its denominator, polynomial division can also be used to reduce the degree instead of repeated application of (2.5).

Engineering problems typically have rational functions with real coefficients. See Exercise 2.2 for a similar technique that decomposes rational functions with real coefficients into terms whose denominators are either powers of linear terms with real zeros or powers of irreducible quadratics with real coefficients.

2.3 Power Series

More complicated functions are found by taking limits of polynomials. Here is the primary example:

$$\sum_{n=0}^{\infty} z^n.$$

This series is important to understand because its behavior is typical of all power series (defined shortly) and because it is one of the few series we can actually add up explicitly.