6 Elementary Maps

In this chapter we will study the mapping properties of the elementary functions and their compositions. The emphasis will be on the behavior of LFTs and the power, trigonometric and exponential functions related to familiar elementary functions of a real variable. These functions are all built from linear functions $a + bz$, $e^z = \sum_0^\infty z^n/n!$ and its locally defined inverse $\log z$ using algebraic operations and composition.

To facilitate our study, we will illustrate these functions using color pictures. Figure 6.1(b) shows a polar grid on the plane, where rays are colored using a standard color wheel in counter-clockwise order beginning along the negative reals: red, yellow, green, cyan, blue, magenta, red. Circles of radius $(1+\varepsilon)^n$, $n = -6, \ldots, 6$, are also plotted using a gray scale, increasing in darkness with the modulus except for the unit circle which is plotted in black with a thicker line width for emphasis. We will call this picture the **standard polar grid**. A "picture" of a complex-valued function f can be created by plotting points z using the same color that $f(z)$ has on the polar grid. For example, the Figure 6.1(a) shows the plot of a rational function. The rational function is a map from Figure 6.1(a) to the polar grid in Figure 6.1(b). Note that the colors near $z = 3$ cycle twice around in the same order as in the polar grid in Figure 6.1(b). This means that there is a zero of order 2 at $z = 3$. The colors near $2i$ and near $-2i$ cycle once in the opposite, or clockwise, order. This means that the function has poles of order one at $\pm 2i$. In fact, it is a picture of the function $(z-3)^2/(z^2+4)$. The preimage of the unit circle is black.

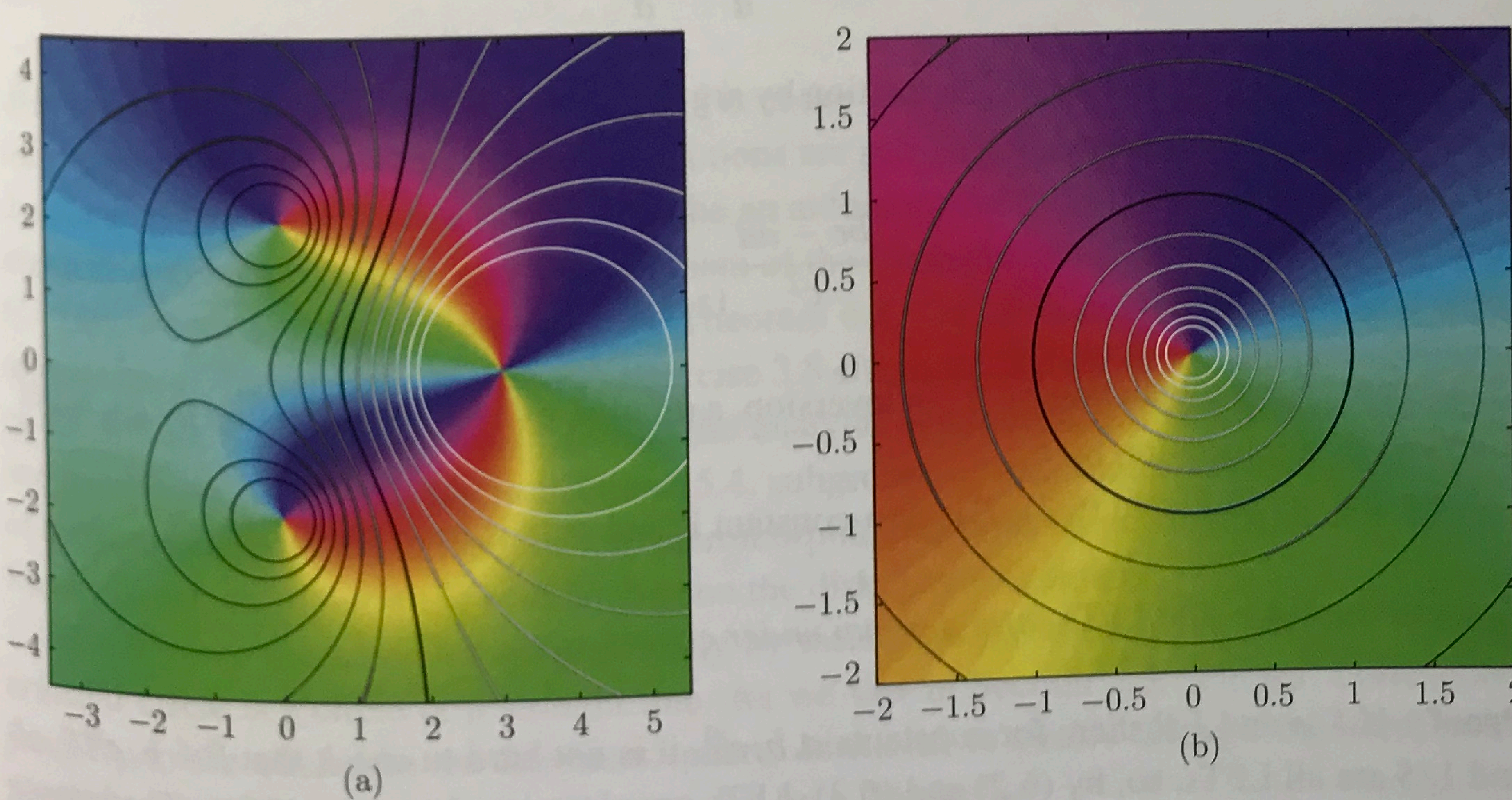

Figure 6.1 A rational function.

See Penrose and Rindler [19] for the connection between Lorentz transformations and LFTs in the theory of relativity.

6.2 Exp and Log

In Exercise 2.6, we encountered the function

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y).$$

This function maps the horizontal line $y = c$ onto the ray $\arg z = c$ from 0 to ∞, and it maps each segment of length 2π in the vertical line $x = c$ onto the circle $|z| = e^c$. Figure 6.3(a) is mapped onto the standard polar grid by the map e^z.

By Exercise 2.7(a), $\frac{d}{dz}e^z = e^z$, which is non-zero, so e^z has a (local) inverse in a neighborhood of each point of $\mathbb{C} \setminus \{0\}$, called $\log z$. As we saw in Corollary 5.8(iii), $\log z$ can be defined as an analytic function on some regions which are not just small disks. For example, the function z is non-zero on the simply-connected region $\mathbb{C} \setminus (-\infty, 0]$. Then, $\log z$, with $\log 1 = 0$, is the function given by

$$\log z = \log |z| + i \arg z, \tag{6.5}$$

where $-\pi < \arg z < \pi$. Figure 6.3(b) is mapped onto the standard polar grid by this function. If instead we specified that $\log 1 = 2\pi i$, then (6.5) holds with $\pi < \arg z < 3\pi$. If $\Omega = \mathbb{C} \setminus (S \cup \{0\})$, where S is the spiral given in polar coordinates by $r = e^\theta$, $-\infty < \theta < \infty$, then Ω is simply-connected and $\operatorname{Im} \log z$ is unbounded on Ω. In this case we can still specify, for example, $\log(-1) = \pi i$, and this uniquely determines the function $\log z$ on Ω.

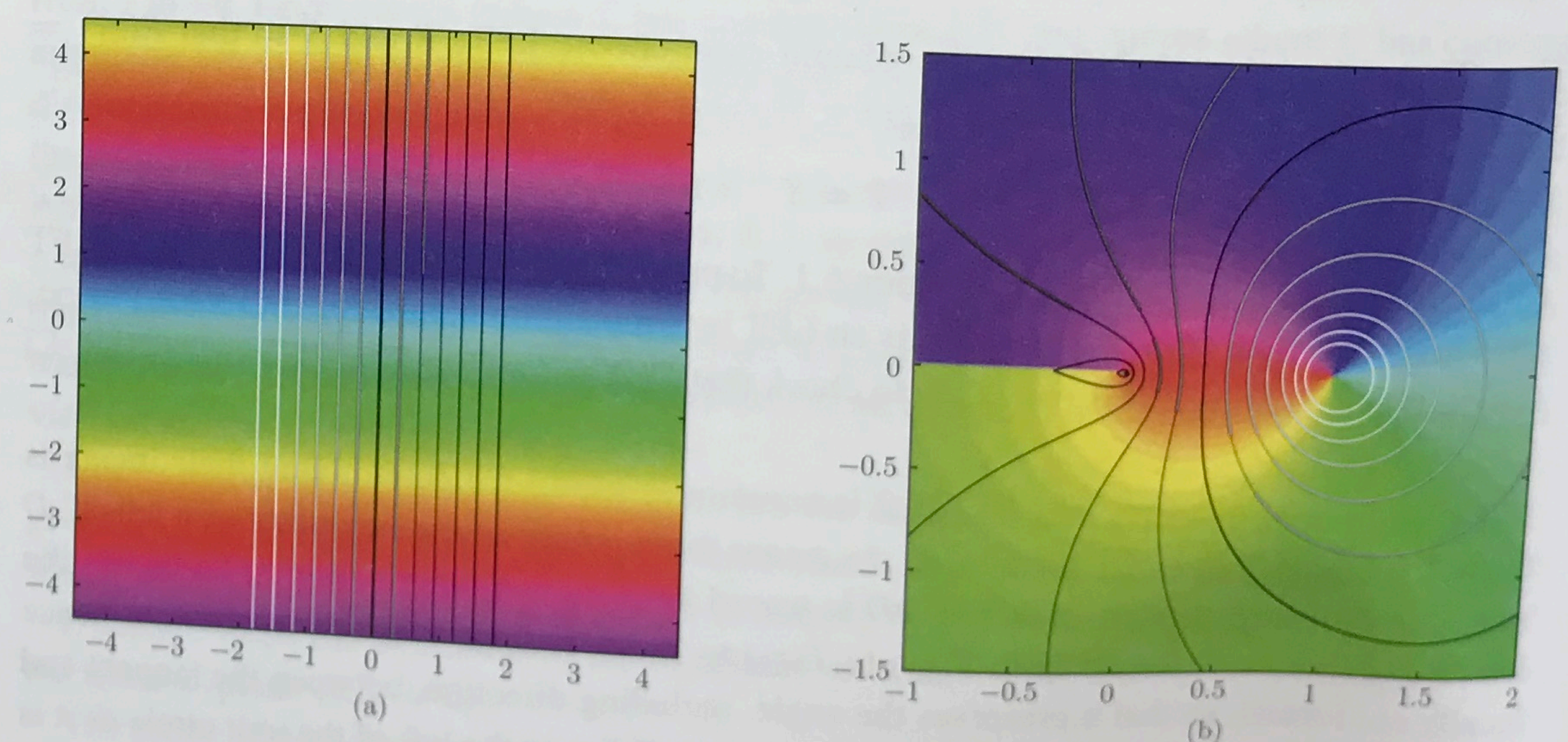

Figure 6.3 Maps e^z and $\log z$.

If Ω is a simply-connected region not containing 0, and if $\alpha \in \mathbb{C}$, we define

$$z^\alpha = e^{\alpha \log z},$$

where $\log z$ can be specified by giving its value at one point $z_0 \in \Omega$. Then z^α is an analytic function on Ω.

For example, suppose $\Omega = \mathbb{C} \setminus (-\infty, 0]$, and define $\log 1 = 0$. If $z = re^{it}$, where $-\pi < t < \pi$, then $z^{1/4} = r^{1/4}e^{it/4}$. Figure 6.4(a) is the preimage of the standard polar grid, slit along $(-\infty, 0]$ by this map. The image of a sector of the form $\{z : |\arg z| < \beta\}$ by this map is the sector $\{z : |\arg z| < \beta/4\}$. Points z on the circle $|z| = r$ are mapped to points on the circle $|z| = r^{1/4}$. The map $z^{1/4}$ is locally conformal in Ω, but it is not conformal at 0. Indeed, angles are multiplied by $1/4$ at 0.

As with the logarithm, it might be easier to understand this map using Figure 6.4(b), which shows the image of the subset $\mathbb{C} \setminus (-\infty, 0]$ of the standard polar grid by this map. Note the range of colors and the location of the level lines. There are four possible definitions of $z^{1/4}$ on $\mathbb{C} \setminus (-\infty, 0]$, depending on the choice of $\log(1) = 2\pi ki$, where k is an integer. Each of the remaining three are rotations of Figure 6.4(b) by integer multiples of $2\pi/4$. If we put

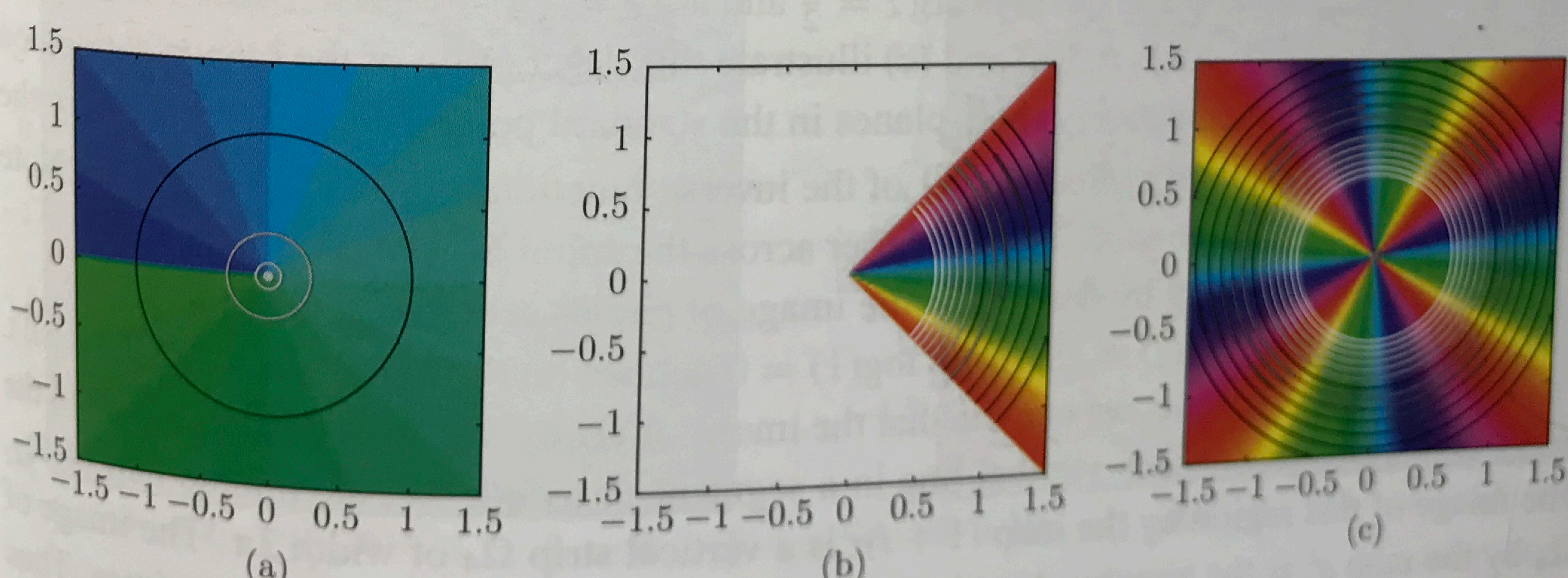

Figure 6.4 Maps z^4 and $z^{1/4}$.

The next function we will consider is

$$w(z) = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

which is analytic in $\mathbb{C} \setminus \{0\}$. See Figure 6.6. By the quadratic formula,

$$z = w \pm \sqrt{w^2 - 1}, \tag{6.6}$$

so that w is two-to-one unless $w^2 = 1$. Since $w(z) = w(1/z)$, the two roots in (6.6) are reciprocals of each other, one inside $\mathbb{D}$ and one outside $\mathbb{D}$, or else complex conjugates of each other on $\partial\mathbb{D}$.

To understand the function w better, we view it as a one-to-one map on various subsets of $\mathbb{C}$. Fix $r > 0$ and write $z = re^{it}$, then

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos t + \frac{i}{2}\left(r - \frac{1}{r}\right)\sin t.$$

If we also write $w = u + iv$, then

$$\left(\frac{u}{\frac{1}{2}(r + \frac{1}{r})}\right)^2 + \left(\frac{v}{\frac{1}{2}(r - \frac{1}{r})}\right)^2 = 1,$$

which is the equation of an ellipse, unless $r = 1$. For each $r \neq 1$, the circles of radius r and $1/r$ are mapped onto the same ellipse. The circle of radius $r = 1$ is mapped onto the interval $[-1, 1]$. We leave as an exercise for the reader to show that the image of a ray from the origin

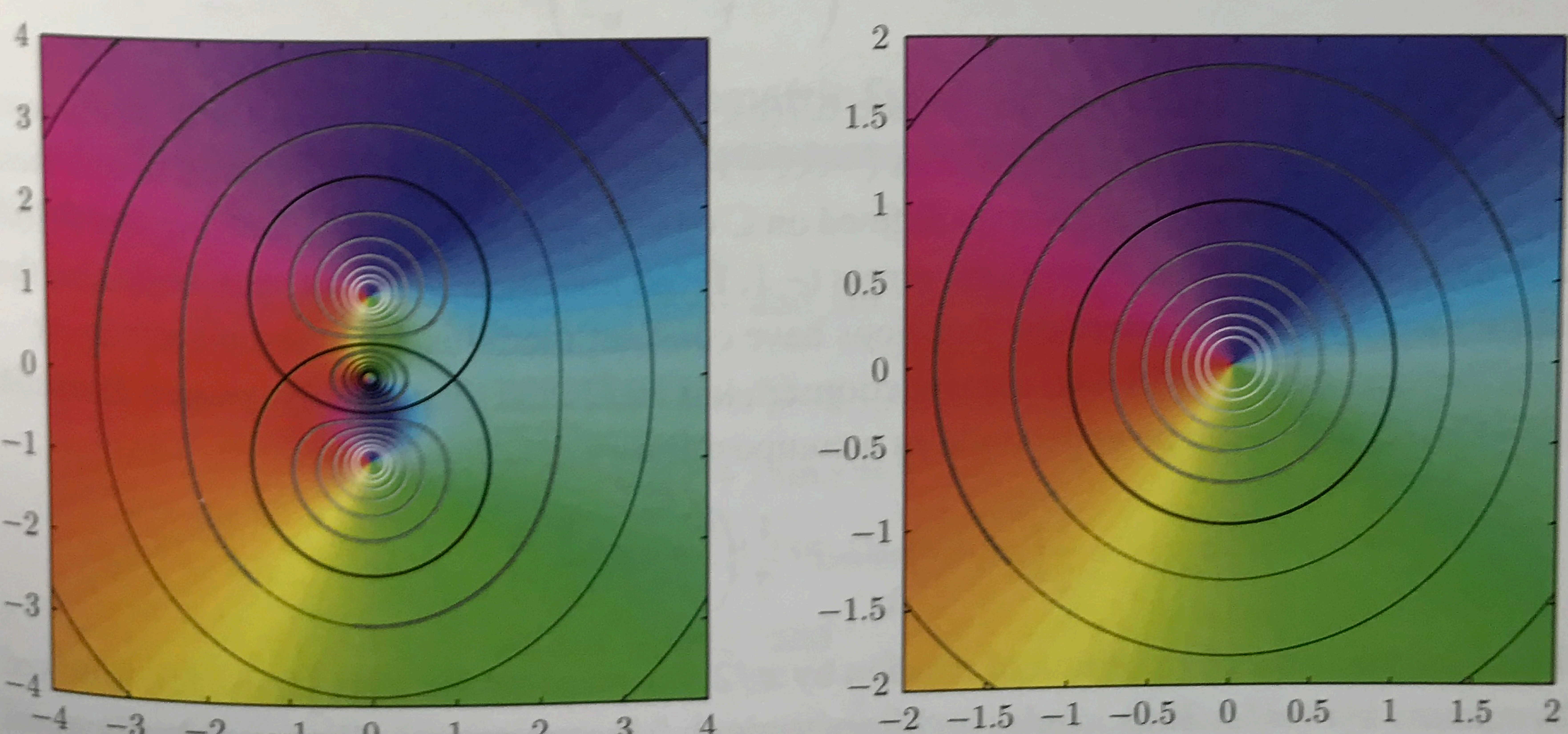

Figure 6.6 The map $(z + 1/z)/2$.

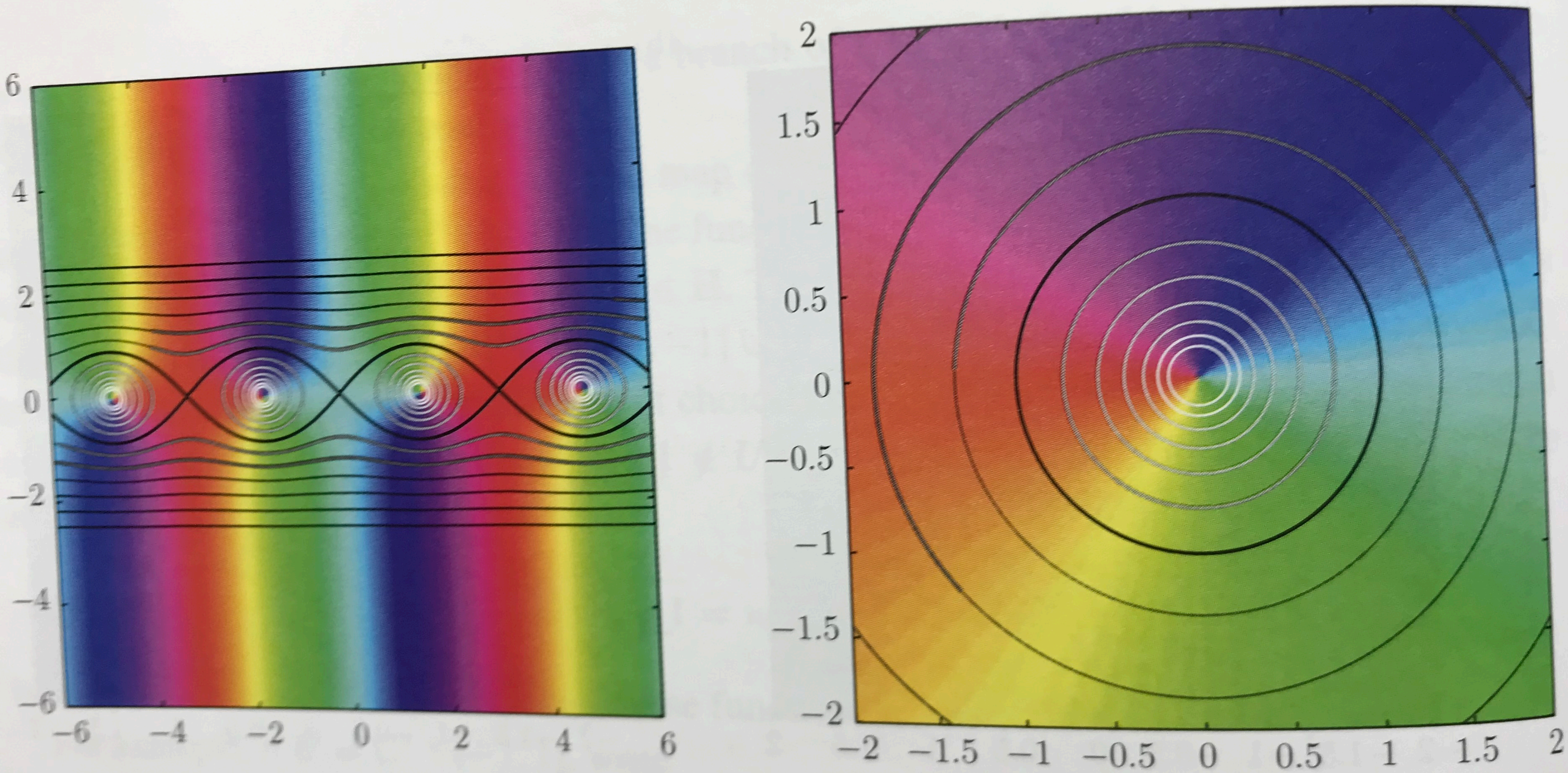

Figure 6.8 The map $\cos(z)$.

the vertical strip $\{z : |\mathrm{Re}z| < \pi\}$ is rotated to the hori...

Figure 6.14 "World's greatest function," $\exp((z+1)/(z-1))$