Midterm 2

Problem 1. (20 points) *The two questions are independent.* Let f be the function given by:

$$f(x,y) = 1 + \sqrt{x^2 - 2xy}$$

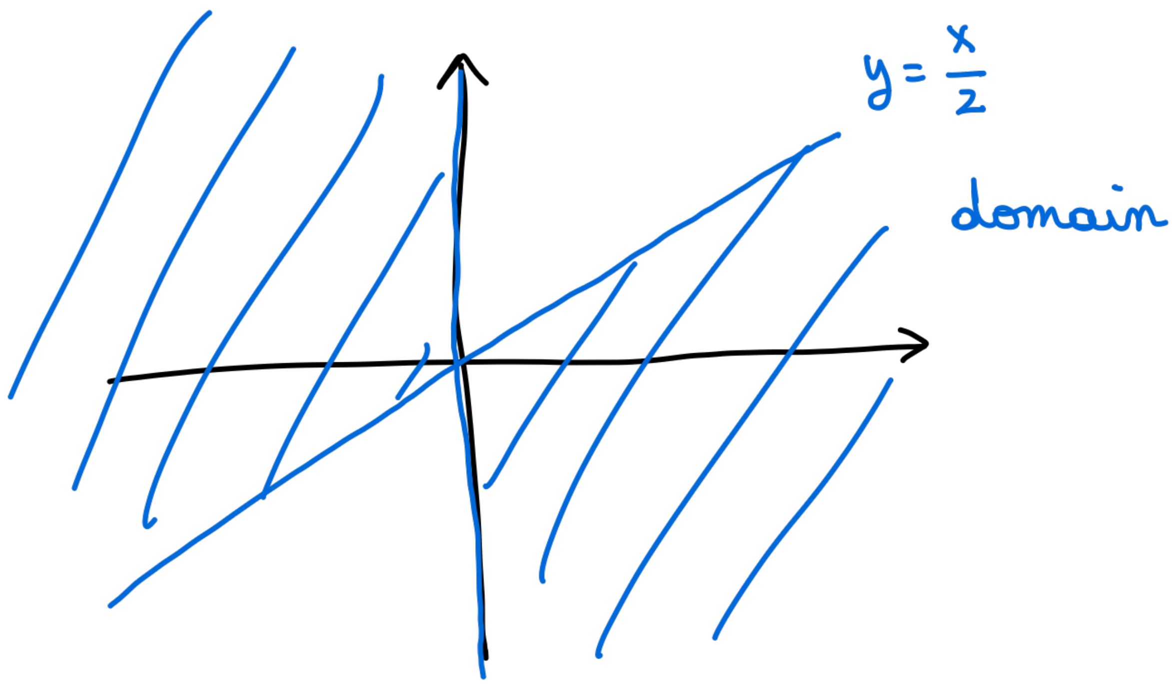
(a) (10 points) Find the domain of f and plot it in the xy-plane.

f is defined when $x^2 - 2xy \geq 0$

$\iff x(x-2y) \geq 0$

$\iff$ "$x \geq 0$ and $x \geq 2y$"

or "$x \leq 0$ and $x \leq 2y$"



(b) (10 points) Find the equation of the tangent plane to the graph of f at the point $(1, 0)$.

$$f_x = \frac{x-y}{\sqrt{x^2-xy}}, \quad f_y = \frac{-x}{\sqrt{x^2-xy}}.$$

At $(1,0)$:

$f_x(1,0) = 1$ $\qquad$ $f_y = -1$

$f(1,0) = 2$

So the equation of this plane is:

$z - 2 = (x-1) - y$

$x - y - z = -1$

Problem 2. (20 points) f be the function given by:

$$f(x,y) = \sqrt{x^2 + 3xy}.$$

Using the linear approximation of f at $(10,10)$, find an approximate value for $f(11,12)$.
You will not get any credit if you compute $f(11,12)$ using your calculator instead of the linear approximation.

$$f(10,10) = \sqrt{100+300} = 20\,;$$

$$f(11,12) \simeq f(10,10) + f_x(10,10)\,(11-10) + f_y(10,10)\,(12-10)$$

$$= 20 + f_x(10,10) + 2f_y(10,10)$$

$$f_x = \frac{2x+3y}{2\sqrt{x^2+3xy}}, \quad f_x(10,10) = \frac{50}{2\cdot 20} = \frac{5}{4}\,;$$

$$f_y = \frac{3x}{2\sqrt{x^2+3xy}}, \quad f_y(10,10) = \frac{30}{2\cdot 20} = \frac{3}{4}.$$

$$f(11,12) \simeq 20 + \frac{5}{4} + 2\cdot\frac{3}{4} = 22.75$$

Problem 3. (20 points) Let f be the function given by:

$$f(x,y) = x^3+y^3-3xy$$

Find all critical points of f and identify whether they correspond to local minima, local maxima or saddle points.

$$\begin{cases} f_x = 3x^2-3y = 0 \\ f_y = 3y^2-3x = 0 \end{cases}$$

$\Leftrightarrow \begin{cases} x^2=y \\ y^2=x \end{cases}$ Hence $x,y \geq 0$ and $x^4=x \Rightarrow x=0$ or $x=1$.

If $x=0$: $y^2=0 \Rightarrow y=0$.

If $x=1$: $x^2=y$, $\Rightarrow y=1$.

Critical points: $(0,0)$ and $(1,1)$.

2nd derivatives:

$f_{xx} = 6x$

$f_{xy} = 3$

$f_{yy} = 6y$.

At $(0,0)$: $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = 3$,

$D = -9$: Saddle point.

At $(1,1)$: $f_{xx} = 6$, $f_{yy} = 6$, $f_{xy} = 3$,

$D = 36 - 9 = 27 > 0$:

local minimum.

Problem 4. (20 points) Let f be the function given by:

$$f(x, y) = x^2 + 2y^2 + 8y$$

Find the global minimum and the global maximum of f in the set

$$S = \{(x, y) \mid x^2 + y^2 \leq 16\}.$$

We test for critical points in the interior of the domain: $x^2+y^2<16$.

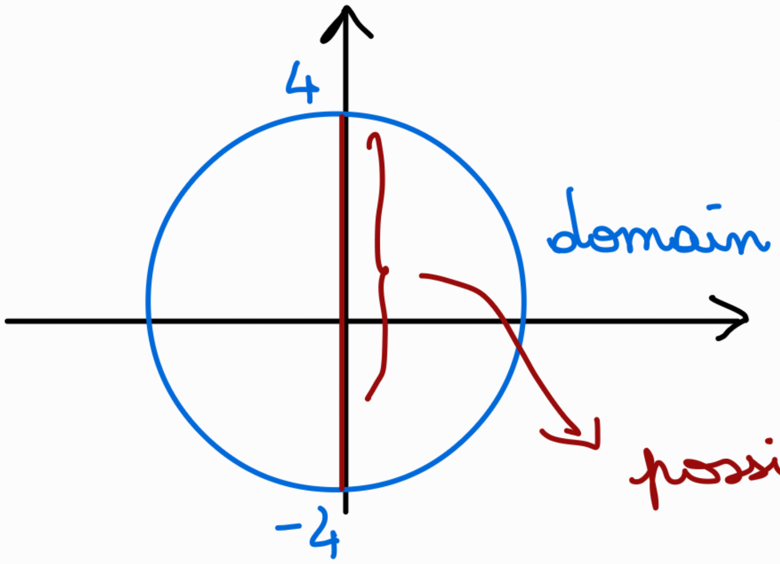
$$f_x = 2x$$
$$f_y = 4y+8$$

$$\Rightarrow \begin{cases} x=0 \\ y=-2 \end{cases}.$$

$$f(0,-2) = 8-16 = -8.$$

Comment: this happens to be a local min, as $f_{xx}=2$, $f_{yy}=4$, $f_{xy}=0$, $D=8>0$, $f_{xx}>0$.

Then we test for points on the boundary, i.e. with $x^2+y^2=16$: circle centered at $(0,0)$ of radius 4



y is between -4 and 4.

With $x^2+y^2=16$,

$$x^2 = 16-y^2,$$

$$\begin{aligned} f(x,y) &= 16-y^2+2y^2+8y \\ &= y^2+8y+16, \\ &= (y+4)^2 = g(y). \end{aligned}$$

Since $g'(y) = 2(y+4) \geq 0$, g is nondecreasing on $[-4,4]$ so its minimum value is $g(-4)=0$ and maximal value is $g(4)=64$.

Since $x^2 = 16-y^2$, we find:

for $y=-4$: $\quad x=0, \quad f(0,-4)=2\cdot 16-32$
$= 0;$

for $y=4$: $\quad x=0, \quad f(0,4)=2\cdot 16+32$
$=64.$

Hence: the minimum value of f is -8 (it is local minimum in the interior and the minimum on the boundary is higher);

and the maximal value is 64 (it is the maximum on the boundary and there are no other local extremum in the interior).

Problem 4. (20 points) *The two questions are independent.* Let $\vec{r}(t)$ be the curve of equation

$$\vec{r}(t) = \left\langle 4t, t^2, t^3 \right\rangle .$$

(a) (10 points) Find a point on this curve whose tangent line is parallel to the line of equation

$$\begin{cases} x = 4t \\ y = -2t \\ z = 3t \end{cases} .$$

The directing vector is $\langle 4, -2, 3 \rangle$.

Moreover, $\vec{r}'(t) = \langle 4, 2t, 3t^2 \rangle$.

So we should have:

$$\begin{cases} 4 = 4 \\ 2t = -2 \\ 3t^2 = 3 \end{cases} \Rightarrow \begin{cases} 4 = 4 \checkmark \\ t = -1 \\ t^2 = 1 \checkmark \end{cases}$$

So $\vec{r}(1) = \langle -4, 1, -1 \rangle$ responds to the problem.

(b) (10 points) Find the curvature of $\vec{r}(t)$ at the point $B(0,0,0)$.

$$\vec{r}'(t) = \langle 4, 2t, 3t^2 \rangle$$

$$\vec{r}''(t) = \langle 0, 2, 6t \rangle$$

$$\vec{r}'(0) = \langle 4, 0, 0 \rangle$$

$$\vec{r}''(0) = \langle 0, 2, 0 \rangle.$$

$$k = \frac{|\vec{r}'(0) \times \vec{r}''(0)|}{|\vec{r}'(0)|^3} = \frac{|\langle 4,0,0\rangle \times \langle 0,2,0\rangle|}{\sqrt{4^2}^3}$$

$$= \frac{8}{4^3} = \frac{8}{64} = \frac{1}{8}.$$