Solutions to Math 126 D Autumn 2024 Midterm II

1. Implicit differentiation:

$$3x^2 = 3yz + 3xyz_x \quad \text{and} \quad 3y^2 = 3xz + 3xyz_y$$

or

$$x^2 = yz + xyz_x \quad \text{and} \quad y^2 = xz + xyz_y$$

at the point $\left(1, 2, \frac{3}{2}\right)$:

$$1 = 2 \cdot \frac{3}{2} + 2 \cdot z_x \quad \text{and} \quad 4 = \frac{3}{2} + 2 \cdot z_y$$

so $z_x = -1$ and $z_y = 5/4$ and the tangent plane is

$$z - \frac{3}{2} = -1(x-1) + \frac{5}{4}(y-2)$$

so

$$1.44 - 1.5 \approx -1(0.85 - 1) + \frac{5}{4}(b-2)$$

and $b \approx 1.832$.

2. If (x, y, x) is the corner of the box in the first octant, then the volume is $V = 4xyz$ so we want to maximize

$$V = 4xy(9 - 4x^2 - y^2) = 36xy - 16x^3y - 4xy^3$$

Taking partials

$$0 = V_x = 36y - 48x^2y - 4y^3 = 4y(9 - 12x^2 - y^2)$$

and

$$0 = V_y = 36x - 16x^3 - 12xy^2 = 4x(9 - 4x^2 - 3y^2)$$

since $x, y > 0$ we must have $9 = 12x^2 + y^2$ and $9 = 4x^2 + 3y^2$ so $x^2 = 9/16$ and $y^2 = 9/4$ which give $x = 3/4$ and $y = 3/2$ giving the volume

$$V = 4 \cdot \frac{3}{4} \cdot \frac{3}{2}\left(9 - 4 \cdot \frac{9}{18} - \frac{9}{4}\right) = \frac{81}{4}$$

To verify maximum:

$$V_{xx} = -96xy, V_{xy} = 36 - 48x^2 - 12y^2, V_{yy} = -24xy$$

at $x = 3/4$ and $y = 3/2$

$$V_{xx} = -108 < 0, V_{xy} = -18, V_{yy} = -27, D = 108(27) - 18^2 = 2592 > 0$$

so we have maximum.

3. Switching the order of integration:

$$\int_0^3 \int_{x^2}^9 \frac{x^3}{\sqrt{x^4 + y^2}}\, dy\, dx = \int_0^9 \int_0^{\sqrt{y}} \frac{x^3}{\sqrt{x^4 + y^2}}\, dx\, dy = \int_0^9 \frac{2}{4}(x^4 + y^2)^{1/2}\bigg|_0^{\sqrt{y}}\, dy$$

$$= \frac{1}{2}\int_0^9 (2y^2)^{1/2} - (y^2)^{1/2}\, dy = \frac{\sqrt{2} - 1}{2}\ \frac{y^2}{2}\bigg|_0^9 = \frac{81\left(\sqrt{2} - 1\right)}{4}$$

4. The volume is

$$\iint_D 1 - x^2 - y^2 \ dA$$

where D is inside the circle $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$. To integrate, we switch to polar:

$$\int_0^\pi \int_0^{\cos\theta} (1 - r^2)\ r\, dr\, d\theta = \int_0^\pi -\frac{(1 - r^2)^2}{4}\bigg|_0^{\cos\theta} d\theta = \int_0^\pi -\frac{(1 - \cos^2\theta)^2}{4} + \frac{1}{4}\, d\theta$$

$$= \frac{1}{4}\int_0^\pi 1 - (\sin^2\theta)^2\,d\theta = \frac{1}{4}\int_0^\pi 1 - \left(\frac{1-\cos 2\theta}{2}\right)^2\,d\theta = \frac{1}{4}\int_0^\pi 1 - \frac{1-2\cos 2\theta+\cos^2 2\theta}{4}\,d\theta$$

$$= \frac{1}{4}\int_0^\pi 1 - \frac{1-2\cos 2\theta + \frac{1+2\cos 4\theta}{2}}{4}\,d\theta = \frac{1}{4}\int_0^\pi \frac{5}{8} - \frac{\cos 2\theta}{2} - \frac{\cos 4\theta}{4}\,d\theta = \frac{1}{4}\left(\frac{5}{8}\theta - \frac{\sin 2\theta}{4} - \frac{\sin 4\theta}{16}\right)\Bigg|_0^\pi = \frac{5\pi}{32}$$