1. (11 pts) Consider the function $f(x, y) = \cos(x)\cos(y)$.

(a) Find all the critical points of the function for $-\frac{\pi}{2} < x < \pi$, $-\frac{\pi}{2} < y < \pi$.

$f_x = -\sin x \cos y = 0 \Rightarrow \sin x = 0$ or $\cos y = 0 \Rightarrow x = 0$ or $y = \frac{\pi}{2}$

$f_y = \cos x(-\sin y) = 0 \Rightarrow \cos x = 0$ or $\sin y = 0 \Rightarrow x = \frac{\pi}{2}$ or $y = 0$

$\Rightarrow$ cr pts: $x=0, y=0$ $(0,0)$

$y = \frac{\pi}{2}, x = \frac{\pi}{2}$ $(\frac{\pi}{2}, \frac{\pi}{2})$

(b) Classify if the function has a local maximum, local minimum, or a saddle point at each critical point you found in (a).

$f_{xx} = -\cos x \cos y$ $f_{xy} = -\sin x(-\sin y) = \sin x \sin y$

$f_{yy} = -\cos x \cos y$

$D = f_{xx} f_{yy} - f_{xy}^2 = \cos^2 x \cos^2 y - \sin^2 x \sin^2 y$

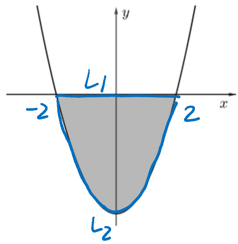
$D(0,0) = 1\cdot 1 - 0\cdot 0 = 1 > 0$, $f_{xx}(0,0) = -1\cdot 1 < 0$

$\Rightarrow$ local max @ $(0,0)$

$D(\frac{\pi}{2}, \frac{\pi}{2}) = 0\cdot 0 - 1\cdot 1 < 0 \Rightarrow$ saddle pt @ $(\frac{\pi}{2}, \frac{\pi}{2})$

2. (11 pts) Consider the function $f(x, y) = x^2y + y^2$

(a) Find the absolute (global) maximum and minimum values of $f$ on the region $D$ enclosed by the parabola $y = x^2 - 4$ and the $x$-axis.




Step 1. Find interior cr pt

$f_x = 2xy = 0 \Rightarrow x = 0$ or $y = 0$

$f_y = x^2 + 2y = 0$

$\Downarrow$

cr pt $(0, 0)$ is on the boundary

Step 2, Check the boundary.

$L_1$: line $y = 0$, $-2 \le x \le 2$

$f(x, 0) = 0$

$L_2$: parabola $y = x^2 - 4 \Rightarrow x^2 = y + 4$ or substitute $y = x^2 - 4$

$f(x, y) = x^2y + y^2 = (y+4)y + y^2$ $\quad f(x, y) = x^2(x^2-4) + (x^2-4)^2$

$g(y) = 2y^2 + 4y \quad y \in [-4, 0]$ $\quad g(x) = 2x^4 - 12x^2 + 16 \quad x \in [-2, 2]$

$g'(y) = 4y + 4 = 0$ $\quad g'(x) = 8x^3 - 24x = 8x(x^2 - 3) = 0$

$x = 0$ or $x = \pm\sqrt{3}$

$\Rightarrow$ cr pt @ $y = -1$, $g(-1) = 2 - 4 = -2$, $f(\pm\sqrt{3}, -1) = -2$ $\quad$ end pts $x = \pm 2$ are on $L_1$

end pt @ $y = -4$, $g(-4) = 32 - 16 = 16$, $f(0, -4) = 16$

$y = 0$, $g(0) = 0$

absolute (global) maximum on $D$: $f(\ 0\ ,\ -4\ ) = 16$

absolute (global) minimum on $D$: $f(\pm\sqrt{3}\ ,\ -1\ ) = -2$

(b) Circle your answer to the following questions, no need to show work or explain.

$f(x, y) = x^2y + y^2 \to +\infty$ if $x \to +\infty$, $y \to +\infty$

Does $f$ attain any absolute (global) maximum on the entire $\mathbb{R}^2$? Yes No

Does $f$ attain any absolute (global) minimum on the entire $\mathbb{R}^2$? Yes No

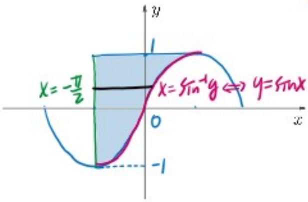
$f(x, y) = y(x^2 + y) \to -\infty$ if $y \to -\infty$, $x^2 + y > 0$

 (or for example, when $y = -1$, $x \to \infty$)

3. (11 pts) Consider the integral

$$\int_{-1}^{1}\int_{-\frac{\pi}{2}}^{\sin^{-1} y} (x+\cos x)^3 \, dx\, dy$$

(a) Sketch and shade the region of integration on the $xy$ plane.

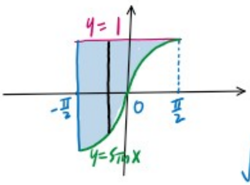



type II: left, right

$$-\frac{\pi}{2} \le x \le \sin^{-1} y$$

$$-1 \le y \le 1$$

(b) Reverse the order of integration and evaluate the integral.




type I: bottom, top

$$\sin x \le y \le 1$$

$$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\sin x}^{1} (x+\cos x)^3 \, dy\, dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x+\cos x)^3 \, y \Big|_{y=\sin x}^{y=1} dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x+\cos x)^3 (1-\sin x)\, dx \qquad u = x+\cos x$$

$$du = (1-\sin x)\,dx$$

$$= \frac{(x+\cos x)^4}{4}\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \qquad \int u^3\, du = \frac{u^4}{4}$$

$$= \frac{1}{4}\Big[\big(\tfrac{\pi}{2} + \underbrace{\cos\tfrac{\pi}{2}}_{0}\big)^4 - \big(-\tfrac{\pi}{2} + \underbrace{\cos(-\tfrac{\pi}{2})}_{0}\big)^4\Big]$$

$$= \frac{1}{4}\Big[\big(\tfrac{\pi}{2}\big)^4 - \big(\tfrac{\pi}{2}\big)^4\Big] = 0$$

4. (12 pts) Part (a) and (b) of this question are independent.

(a) Setup a double integral in Polar coordinate to find the volume of the solid enclosed by the hyperboloid $-x^2 - y^2 + 2z^2 = 4$ and the cone $z = \sqrt{x^2 + y^2}$.
Do **NOT** evaluate the integral.

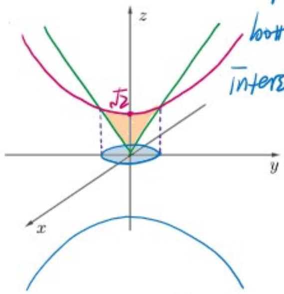
$$\int_{\boxed{0}}^{\boxed{2\pi}} \int_{\boxed{0}}^{\boxed{2}} \boxed{\left(\sqrt{\frac{4+r^2}{2}} - r\right) r} \, d\boxed{r} \, d\boxed{\theta}$$

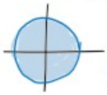
top: $-x^2-y^2+2z^2=4 \rightsquigarrow -r^2+2z^2=4 \Rightarrow z=\sqrt{\frac{4+r^2}{2}}$

bottom: $z=\sqrt{x^2+y^2} \rightsquigarrow z=r \ (r>0)$

intersection: $\sqrt{\frac{4+r^2}{2}}=r \Rightarrow \frac{4+r^2}{2}=r^2$

$\Rightarrow r^2=4 \Rightarrow r=2 \ (r \geq 0)$



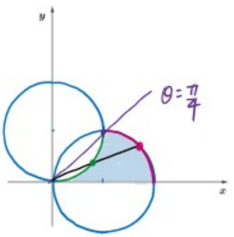



D: $0 \leq r \leq 2$
$0 \leq \theta \leq 2\pi$

*You do not have to sketch the 3D graph.*

(b) Setup the integral $\iint_D y \, dA$ in Polar coordinate, $D$ is the region **in the first quadrant** that lies inside the circle $x^2 + y^2 = 4x$ and outside the circle $x^2 + y^2 = 4y$.
Do **NOT** evaluate the integral.

$$\int_{\boxed{0}}^{\boxed{\frac{\pi}{4}}} \int_{\boxed{4\sin\theta}}^{\boxed{4\cos\theta}} \boxed{r\sin\theta \ r} \, d\boxed{r} \, d\boxed{\theta}$$




Inner: $x^2+y^2=4y \rightsquigarrow r=4\sin\theta$

outer: $x^2+y^2=4x \rightsquigarrow r=4\cos\theta$

intersection: $4\sin\theta=4\cos\theta \Rightarrow \frac{\sin\theta}{\cos\theta}=1 \Rightarrow \theta=\frac{\pi}{4}$

D: $4\sin\theta \leq r \leq 4\cos\theta$
$0 \leq \theta \leq \frac{\pi}{4}$