Problem 1. (a) (10 points) Show that the points $A(0,0,1)$ and $B(1,2,3)$ belong to the curve of equation

$$\vec{r}(t) = \left\langle t^3, 2t^3, 2t^3+1 \right\rangle.$$

- To check that A belongs to this curve, it suffices to verify that $\vec{r}(t) = \langle 0,0,1\rangle$ for some t. We have:

$$\vec{r}(t) = \langle 0,0,1\rangle \iff \begin{cases} t^3 = 0 \\ 2t^3 = 0 \\ 2t^3+1 = 1 \end{cases} \iff \begin{cases} t=0 \\ t=0 \\ t=0 \end{cases} \checkmark$$

so $\vec{r}(0) = \langle 0,0,1\rangle$: A belongs to this curve.

- Likewise, we solve $\vec{r}(t) = \langle 1,2,3\rangle$:

$$\iff \begin{cases} t^3 = 1 \\ 2t^3 = 2 \\ 2t^3+1 = 3 \end{cases} \iff \begin{cases} t=1 \\ t=1 \\ t=1 \end{cases} \checkmark$$

so $\vec{r}(1) = \langle 1,2,3\rangle$: B belongs to this curve.

<u>Comment:</u> you get full credit for simply noticing $\vec{r}(0) = \langle 0,0,1\rangle$
$\vec{r}(1) = \langle 1,2,3\rangle$.

(b) (10 points) Find the length of this curve between $A(0,0,1)$ and $B(1,2,3)$.

Since A corresponds to $t=0$ and B corresponds to $t=1$, this length is

$$\int_0^1 |\vec{r}\,'(t)|\,dt.$$

$$\vec{r}\,'(t) = \langle 3t^2, 6t^2, 6t^2 \rangle$$

$$|r'(t)| = \sqrt{(3t^2)^2 + (6t^2)^2 + (6t)^2}$$

$$= \sqrt{81 \cdot t^4} = 9t^2$$

So $\int_0^1 |r'(t)|\,dt = \int_0^1 9t^2 \cdot dt = 9 \cdot \frac{t^3}{3}\Big|_0^1 = 3$

The length is 3.

Comment: you could have also noticed that the graph of $\vec{r}(t)$ is a line:

with $s = t^3$, $r(t) = \langle s, 2s, 2s+1 \rangle$! so, the length of that arc is $|AB| = \sqrt{1^2+2^2+2^2} = 3$.

Problem 2. (20 points) Find the intersection of the two lines of equation:

$$\begin{cases} x = 2 + t \\ y = 3 - t \\ z = 1 - 2t \end{cases} \quad \text{and} \quad \begin{cases} x = 1 + t \\ y = 1 + 2t \\ z = 4 - 3t \end{cases}.$$

Hint: be careful about setting the right equation!

These are parametric lines! to find the intersection of the two graphs, we must use <u>independent parameters</u>:

that is, solve:

$$\begin{cases} 2+t = 1+s \\ 3-t = 1+2s \\ 1-2t = 4-3s \end{cases} \iff \begin{cases} s = 1+t \\ 3-t = 1+2(1+t) \\ 1-2t = 4-3(1+t) \end{cases}$$

$$\iff \begin{cases} s = 1+t \\ -t = 2t \\ -2t = -3t \end{cases} \iff \begin{cases} s = 1 \\ t = 0 \\ t = 0 \end{cases}.$$

So, these two lines intersect at

$(x,y,z) = (2,3,1)$.

Problem 3. (a) (10 points) Find a vector-valued function whose graph is the intersection of the surfaces

$$\overset{x^2}{\cancel{(x-1)^2}} + (y+2)^2 = 9 \quad \text{and} \quad x^{\cancel{2}} - z = 1.$$

We use $(3\cos t)^2 + (3\sin t)^2 = 9$

to set: $\begin{cases} x = 3\cos t \\ y+2 = 3\sin t \\ z = x-1 \end{cases}$

$$\Rightarrow \begin{cases} x = 3\cos t \\ y = -2+3\sin t \\ z = -1+3\cos t \end{cases}$$

(b) (10 points) Find the equation of the line that, at $t = 0$, is tangent to the graph of the vector-valued function

$$\vec{r}(t) = \langle -t, \cos(2t), 3t^2 \cancel{\rangle .} + 3t \rangle$$

We have $\vec{r}(0) = \langle 0, 1, 0 \rangle$. This point must be on the tangent line.

Moreover, the direction of the tangent line is $\vec{r}\,'(0)$. Since

$$\vec{r}\,'(t) = \langle -1, -2\sin(2t), 6t+3 \rangle,$$

$$\vec{r}\,'(0) = \langle -1, 0, 3 \rangle$$

We conclude that this tangent line has equation:

$$\begin{cases} x = 0 - 1s \\ y = 1 - 0s \\ z = 0 + 3s \end{cases} \Leftrightarrow \begin{cases} x = -s \\ y = 1 \\ z = 3s. \end{cases}$$

Problem 4. (a) Find the intersection of the plane of equation $x - 2y + z = 0$ with the line

$$\begin{cases} x = 3 - t \\ y = 1 + t \\ z = -1 + 2t \end{cases}$$

We first solve:

$$\begin{cases} x - 2y + z = 0 \\ x = 3 - t \\ y = 1 + t \\ z = -1 + 2t \end{cases} \Rightarrow (3-t) - 2(1+t) + (-1+2t) = 0$$

$$\Rightarrow 0 - t = 0 \Rightarrow t = 0.$$

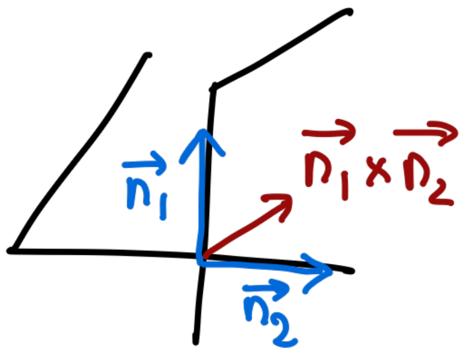
So, $\begin{cases} x = 3 \\ y = 1 \\ z = -1 \end{cases}$.

Hence, this line and plane intersect at the point of coordinates $(3, 1, -1)$.

Problem 4 (20 points)

[redacted] Find an equation for the intersection of the two planes given by the equations $2x+y-z=1$ and $x+3y-z=0$.

These two planes have orthogonal vector $\vec{n}_1 = \langle 2,1,-1 \rangle$ and $\vec{n}_2 = \langle 1,3,-1 \rangle$.



We have:

$$\vec{n}_1 \times \vec{n}_2 = \langle 2,1,-1 \rangle \times \langle 1,3,-1 \rangle$$

$$= \langle 2,1,5 \rangle.$$

This is a non-zero vector, hence the two planes are not parallel. They must intersect along a line directed by $\vec{n}_1 \times \vec{n}_2$. It remains to find a point in the intersection.

We have: $\begin{cases} 2x+y-z=1 \\ x+3y-z=0 \end{cases}$

$$\Leftrightarrow \begin{cases} x-2y=1 \\ x+3y-z=0 \end{cases} \Leftrightarrow \begin{cases} x=1+2y \\ 1+5y-z=0 \end{cases}$$

So $z=1$, $y=0$, $x=1$ works: $(1,0,1)$ is in the intersection. So a parametric equation for this line is:

$$\begin{cases} x = 1+2t \\ y = t \\ z = 1+5t. \end{cases}$$

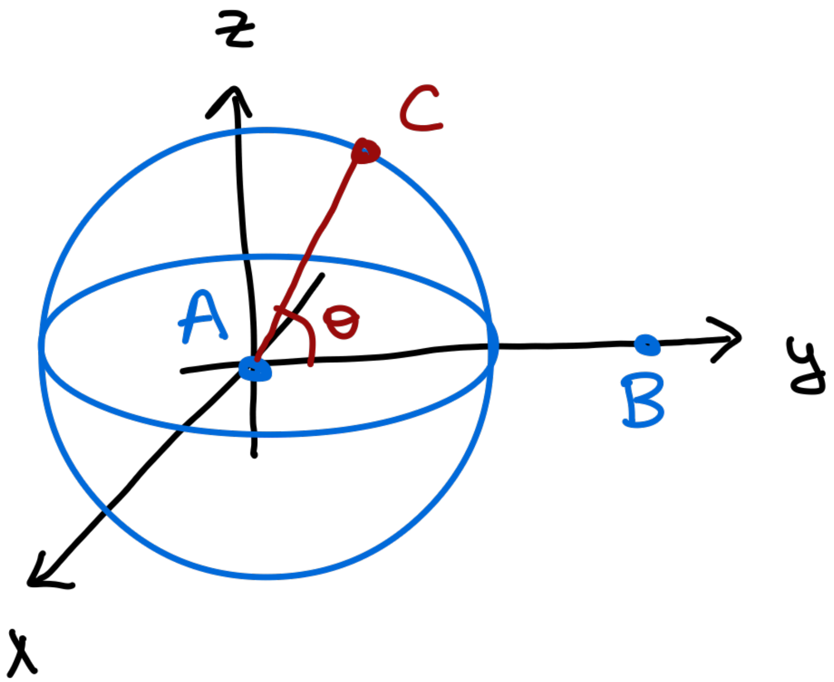
Sanity check:

$$2(1+2t)+t-(1+5t) = 2+5t-1-5t = 1 \quad \checkmark$$

$$(1+2t)+3t-(1+5t) = 1-1+5t-5t = 0 \quad \checkmark$$

So this line is indeed in both planes!

Problem 5. Let $A(0,0,0)$ and $B(0,2,0)$ and S be the sphere of radius 1, centered at A. Find an equation for the set of all points C on the sphere S that maximize the area of the triangle ABC.



The area of ABC is:

$$\mathcal{A} = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{|AB| \cdot |AC| \cdot |\sin\theta|}{2}$$

where $\sin\theta$ is the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. We have:

$|AB| = 2$, $|AC| = 1$ (because C is on the sphere of radius 1, centered at A).

So, $\mathcal{A} = |\sin\theta|$.

This is maximal for $\theta = \pi/2$, so when $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are orthogonal. Since $\overrightarrow{AB}$ points in the y-direction, C must be on the xz plane, on top of being on $\mathcal{S}$.

So, if C has coordinates (x,y,z) then
$$\begin{cases} x^2+y^2+z^2=1 \\ y=0 \end{cases}$$

$$\Rightarrow \begin{cases} x^2+z^2=1 \\ y=0 \end{cases}$$

Comment: this is the circle in the xz-plane, centered at A, of radius 1.