1. (13 pts)

(a) Find a vector that has length 7 and is orthogonal to both $\mathbf{u} = \langle 1, 0, 2 \rangle$ and $\mathbf{v} = \langle 3, -2, 1 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 3 & -2 & 1 \end{vmatrix} = (0 - -4)\vec{i} - (1-6)\vec{j} + (-2-0)\vec{k} = \langle 4, 5, -2 \rangle$$

CHECK: $4 + 0 - 2 = 0$ ✓
$12 - 10 - 2 = 0$ ✓

$$|\vec{u} \times \vec{v}| = \sqrt{16+25+4} = \sqrt{45} = 3\sqrt{5}$$

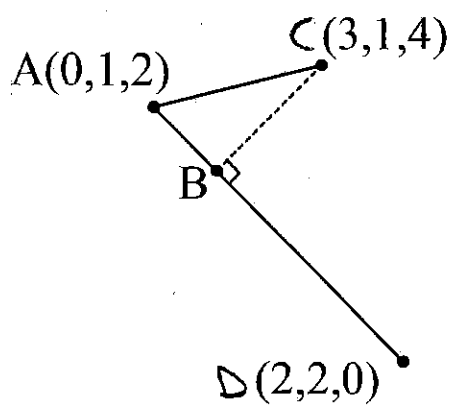
$$\boxed{\frac{7}{\sqrt{45}} \langle 4, 5, -2 \rangle} \quad \text{OR} \quad \boxed{\frac{-7}{\sqrt{45}} \langle 4, 5, -2 \rangle}$$

(b) Find the distance from point A to point B in the picture below (Hint: Use vector tools!)

$\vec{AC} = \langle 3, 0, 2 \rangle$

$\vec{AD} = \langle 2, 1, -2 \rangle$

$$\text{COMP}_{\vec{AD}} \vec{AC} = \frac{6+0-4}{\sqrt{4+1+4}} = \boxed{\frac{2}{3}}$$



(c) Consider the line through the points $(0, 0, 1)$ and $(3, 4, 5)$. Find the (x, y, z) point(s) where the line intersects the cylinder $x^2 + y^2 = 4$.

LINE: $x = 0 + 3t, \; y = 0 + 4t, \; z = 1 + 4t$

INTERSECTION: $(3t)^2 + (4t)^2 = 4 \Rightarrow 25t^2 = 4 \Rightarrow t^2 = \frac{4}{25}$

$\Rightarrow t = \pm \frac{2}{5}$

$t = -\frac{2}{5} \Rightarrow \boxed{(x,y,z) = \left(-\frac{6}{5}, -\frac{8}{5}, 1 - \frac{8}{5}\right) = \left(-\frac{6}{5}, -\frac{8}{5}, -\frac{3}{5}\right)}$

$t = \frac{2}{5} \Rightarrow \boxed{(x,y,z) = \left(\frac{6}{5}, \frac{8}{5}, 1 + \frac{8}{5}\right) = \left(\frac{6}{5}, \frac{8}{5}, \frac{13}{5}\right)}$

2. (12 pts)

(a) Find parametric equations for the line of intersection of the planes $x + y + z = 10$ and $x - 3y - 4z = -10$.

TWO POINTS? COMBINING $\Rightarrow$ $4y + 5z = 20$ $\quad z = 10 - x - y$

$y = 0 \Rightarrow z = 4 \Rightarrow x = 6$ $\quad P(6, 0, 4)$ CHECK ✓

$z = 0 \Rightarrow y = 5 \Rightarrow x = 5$ $\quad Q(5, 5, 0)$

$$\boxed{\begin{aligned} x &= 6 - t \\ y &= 0 + 5t \\ z &= 4 - 4t \end{aligned}}$$

ANY POINT ON LINE

ANY VECTOR PARALLEL TO $\langle -1, 5, -4 \rangle$

(b) Consider the plane that passes thru $(4, 4, 2)$ and contains the line $x = 5t, y = 3 + t, z = 4 - t$. Find the (x, y, z) point where this plane intersects the y-axis.

THREE POINTS? $\quad P(0, 3, 4)$, $Q(5, 4, 3)$, $R(4, 4, 2)$

$t = 0 \qquad t = 1$

TWO VECTORS PARALLEL TO DESIRED PLANE $\left\{ \begin{array}{l} \overrightarrow{PQ} = \langle 5, 1, -1 \rangle = \text{SAME AS DIRECTION VECTOR FOR LINE} \\ \overrightarrow{PR} = \langle 4, 1, -2 \rangle \end{array} \right.$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (-2 - -1)\vec{i} - (-10 - -4)\vec{j} + (5 - 4)\vec{k}$$

$$= \langle -1, 6, 1 \rangle$$

CHECK: $-5 + 6 - 1 = 0$ ✓ $\quad -4 + 6 - 2 = 0$ ✓

PLANE: $-(x - 4) + 6(y - 4) + (z - 2) = 0$

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y-AXIS $\Leftrightarrow$ $x = 0$ AND $z = 0$ $\Leftrightarrow$ $4 + 6(y - 4) - 2 = 0$

$6(y - 4) = -2$

$y - 4 = -1/3$

$y = 4 - 1/3 = \frac{11}{3} = 3.\overline{6}$

$$\boxed{\left(0, \tfrac{11}{3}, 0\right)} = (0, 3.\overline{6}, 0)$$

3. (12 pts)

(a) Give the precise 3D name for $4x^2 = 5y^2 + z$.

$z = 4x^2 - 5y^2$

$\boxed{\text{HYPERBOLIC PARABOLOID}}$

(b) Set up, but DO NOT EVALUATE, an integral that represents the arc length of the curve of intersection of the cylinder $x^2 + y^2 = 25$ and $x + 2y + z = 10$.

$x = 5\cos(t)$

$y = 5\sin(t)$

$z = 10 - 5\cos(t) - 10\sin(t)$

$z = 10 - x - 2y$

MANY OTHER PARAMETERIZATIONS ARE VALID

$$\boxed{\int_0^{2\pi} \sqrt{(-5\sin(t))^2 + (5\cos(t))^2 + (5\sin(t) - 10\cos(t))^2}\, dt}$$

$$= \int_0^{2\pi} \sqrt{25 + (5\sin(t) - 10\cos(t))^2}\, dt$$

(c) Consider the curves $\mathbf{r}_1(t) = \langle 2t, 3t^2, t^3 \rangle$ and $\mathbf{r}_2(u) = \langle 2 - 2u, 3 + 3u, u^2 + 1 \rangle$. The curves have one point of intersection. Find the angle of intersection to the nearest degree.

(i) $2t = 2 - 2u \Rightarrow t = 1 - u$

(ii) $3t^2 = 3 + 3u \Rightarrow t^2 = 1 + u \Rightarrow (1-u)^2 = 1 + u$

$\cancel{1} - 2u + u^2 = \cancel{1} + u$

$u^2 - 3u = 0$

$u(u - 3) = 0$

$u = 0$ or $u = 3$

$\Downarrow \quad \Downarrow$

$t = 1 \quad t = -2$

$u = 0, t = 1 \Rightarrow$ (iii) $t^3 = 1$, $u^2 + 1 = 1$ YES!

$u = 3, t = -2 \Rightarrow$ (iii) $t^3 = -8$, $u^2 + 1 = 10$ NO

$\vec{r}_1'(t) = \langle 2, 6t, 3t^2 \rangle \quad \vec{r}_1'(1) = \langle 2, 6, 3 \rangle$

$\vec{r}_2'(u) = \langle -2, 3, 2u \rangle \quad \vec{r}_2'(0) = \langle -2, 3, 0 \rangle$

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{-4 + 18 + 0}{\sqrt{4+36+9}\sqrt{4+9+0}} = \frac{14}{7\sqrt{13}} = \frac{2}{\sqrt{13}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{13}}\right) \approx 56.31^\circ \quad \boxed{56^\circ}$$

4. (13 pts)

(a) Give parametric equations for the tangent line to $\mathbf{p}(t) = \langle t^2, 3-3t, 3+2t \rangle$ at $t = 1$.

$\vec{p}\,'(t) = \langle 2t, -3, 2 \rangle$

$\vec{p}(1) = \langle 1, 0, 5 \rangle$

$\vec{p}\,'(1) = \langle 2, -3, 2 \rangle$

$$\boxed{\begin{aligned} x &= 1+2u \\ y &= 0-3u \\ z &= 5+2u \end{aligned}}$$

(b) Find the principal unit normal vector $\mathbf{N}(t)$ for $\mathbf{q}(t) = \langle 3t, \cos(4t), \sin(4t) \rangle$.

$\vec{q}\,'(t) = \langle 3, -4\sin(4t), 4\cos(4t) \rangle$

$|\vec{q}\,'(t)| = \sqrt{9 + 16\sin^2(4t) + 16\cos^2(4t)} = \sqrt{9+16} = 5$

$\vec{T}(t) = \langle \frac{3}{5}, -\frac{4}{5}\sin(4t), \frac{4}{5}\cos(4t) \rangle$

$\vec{T}\,'(t) = \langle 0, -\frac{16}{5}\cos(4t), -\frac{16}{5}\sin(4t) \rangle$

$|\vec{T}\,'(t)| = \sqrt{\left(\frac{16}{5}\right)^2 \cos^2(4t) + \left(\frac{16}{5}\right)^2 \sin^2(4t)} = \frac{16}{5}$

$$\boxed{\vec{N}(t) = \langle 0, -\cos(4t), -\sin(4t) \rangle}$$

(c) An object is moving such that its velocity is given by $\mathbf{r}'(t) = \langle t, \sin(t), t\cos(t) \rangle$ and its initial location is $\mathbf{r}(0) = \langle 0, 0, 1 \rangle$. Find the position function $\mathbf{r}(t)$.

$\vec{r}(t) = \int \langle t, \sin(t), t\cos(t) \rangle dt$

$= \langle \frac{1}{2}t^2 + c_1, -\cos(t) + c_2, \qquad \rangle$

$\int t\cos(t)\,dt = t\sin(t) - \int \sin(t)\,dt$
$= t\sin(t) + \cos(t) + c_3$

$u = t \quad dv = \cos(t)$
$du = dt \quad v = \sin(t)$

$\vec{r}(0) = \langle 0,0,1 \rangle = \langle c_1, -1+c_2, 1+c_3 \rangle \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 1 \\ c_3 = 0 \end{cases}$

$$\boxed{\vec{r}(t) = \langle \frac{1}{2}t^2, -\cos(t)+1, t\sin(t) + \cos(t) \rangle}$$