1. (12 points)

6 (a) Find the equation of the plane that contains the point $(1, -2, 3)$ and the line given by $x = 4t$, $y = 1 - t$, $z = 5 + 2t$.

NEED TWO VECTORS PARALLEL TO THE PLANE.

POINTS ON PLANE: $\underset{t=0}{A(0,1,5)}$, $\underset{t=1}{B(4,0,7)}$, $\underset{\text{given}}{C(1,-2,3)}$

VECTORS: $\underbrace{\vec{AB} = \langle 4,-1,2\rangle}_{\text{direction vector for line}}$, $\vec{AC} = \langle 1,-3,-2\rangle$

NORMAL: $\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & 2 \\ 1 & -3 & -2 \end{vmatrix} = (2--6)\vec{i} - (-8-2)\vec{j} + (-12--1)\vec{k}$

$= \langle 8, 10, -11\rangle$

PLANE: $\boxed{\begin{aligned} &8(x-1) + 10(y+2) - 11(z-3) = 0 \\ &8x + 10y - 11z + 45 = 0 \end{aligned}}$

6 (b) Consider the line through the point $(0, 3, 5)$ that is orthogonal to the plane $2x - y + z = 20$. Find the point of intersection of the line and the plane.
(Hint: Start by finding parametric equations for the line).

LINE EQUATIONS:
$x = 0 + 2t$
$y = 3 - t$
$z = 5 + t$

NORMAL TO PLANE: $\underbrace{\langle 2,-1,1\rangle}_{\text{DIRECTION VECTOR FOR THE LINE}}$

INTERSECTION: $2x - y + z = 20$

$2(2t) - (3-t) + (5+t) = 20$

$\Rightarrow 4t - 3 + t + 5 + t = 20$

$6t = 18$

$t = 3$

$\Rightarrow \boxed{(x,y,z) = (6,0,8)}$

ASIDE: You could now find the dist. from $(0,3,5)$ to $(6,0,8)$ to get the distance to the plane.

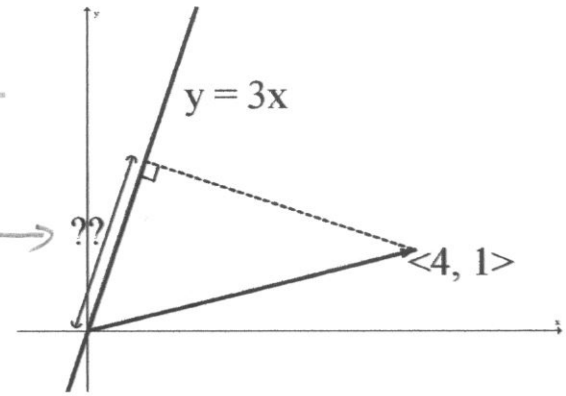
2. (12 points)

$\boxed{5}$ (a) The vector $\langle 4, 1\rangle$ represents the force due to a heavy wind on the xy-plane. Find the length of the projection of this wind onto the line $y = 3x$.
That is, find the indicated length in the picture below:

MANY SOL'NS. Here's one using the projection. $\text{comp}_{\vec{a}}(\vec{b}) = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

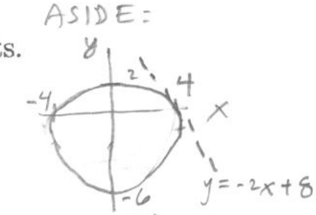
$\vec{a} = \langle 1, 3\rangle \quad \vec{b} = \langle 4, 1\rangle$

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{4+3}{\sqrt{1+9}} = \boxed{\frac{7}{\sqrt{10}}}$$



(b) The polar curve $r = 4 - 2\sin(\theta)$ has exactly two x-intercepts and two y-intercepts.

ASIDE:



$\boxed{2}$ i. Give the (x, y) coordinates for all the intercepts (fill in the blanks).

$\theta = 0 \Rightarrow r = 4 \Rightarrow (x,y) = (4,0)$
$\theta = \pi \Rightarrow r = 4 \Rightarrow (x,y) = (-4,0)$
$\theta = \pi/2 \Rightarrow r = 2 \Rightarrow (x,y) = (0,2)$
$\theta = 3\pi/2 \Rightarrow r = 6 \Rightarrow (x,y) = (0,-6)$

x-intercepts: $(-4, 0)$ and $(4, 0)$.

y-intercepts: $(0, -6)$ and $(0, 2)$.

$\boxed{5}$ ii. Find the equation for the tangent line at the positive x-intercept.

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{(-2\cos\theta)\sin\theta + (4-2\sin\theta)\cos\theta}{(-2\cos\theta)\cos\theta - (4-2\sin\theta)\sin\theta}$$

$\theta = 0 \qquad \dfrac{dr}{d\theta} = -2\cos\theta\,\Big|_{\theta=0} = -2$

$r(0) = 4$

$$\Rightarrow \frac{dy}{dx}\Big|_{\theta=0} = \frac{(-2)(0) + (4)(1)}{(-2)(1) - (4)(0)} = \frac{4}{-2} = -2$$

$(x,y) = (4,0)$

$$\boxed{y = -2(x-4) + 0 = -2x + 8}$$

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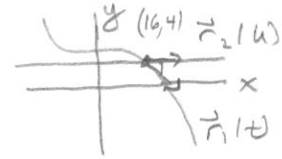
3. ([redacted] points) Consider the two curves given by the position vector functions $\mathbf{r}_1(t) = \langle t^2 + 6t, 12 - t^3 \rangle$ and $\mathbf{r}_2(u) = \langle 2u - 6, 4 \rangle$

ASIDE:



4 (a) Find the equation of the tangent line to the curve given by $\mathbf{r}_1(t)$ at $t = 1$. (Give your final answer into the form $y = mx + b$)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3t^2}{2t+6} \quad \text{at } t=1 \quad \frac{dy}{dx} = \frac{-3}{8}$$

$$x = 7$$
$$y = 11$$

$$\boxed{y = -\frac{3}{8}(x-7) + 11} = -\frac{3}{8}x + \frac{109}{8} = -0.375x + 13.625$$

4 (b) Find a vector $\mathbf{v} = \langle v_1, v_2 \rangle$ that has length 7 and is orthogonal to the tangent vector to $\mathbf{r}_2(u)$ at $u = 4$.

$$\vec{r}_2{}'(u) = \langle 2, 0 \rangle \Rightarrow \vec{r}_2{}'(4) = \langle 2, 0 \rangle$$

ORTHOGONAL $\Rightarrow \langle 2,0 \rangle \cdot \langle v_1, v_2 \rangle = 0 \Rightarrow 2v_1 = 0 \Rightarrow v_1 = 0$

LENGTH 7 $\Rightarrow |\vec{v}| = |\langle 0, v_2 \rangle| = \sqrt{0^2 + v_2^2} = 7$

$\Rightarrow v_2 = 7$ or -7 TWO ANSWERS $\boxed{\langle 0, 7 \rangle}$ or $\boxed{\langle 0, -7 \rangle}$ Either one

5 (c) The two curves have one point of intersection. Find the (acute) angle of intersection between the curves at this point. (Round your final answer to the nearest degree).

① $t^2 + 6t = 2u - 6$

② $12 - t^3 = 4 \Rightarrow 8 = t^3 \Rightarrow t = 2$

① AND ② $\Rightarrow (2)^2 + 6(2) = 2u - 6 \Rightarrow 16 = 2u - 6 \Rightarrow 22 = 2u \Rightarrow u = 11$

$\boxed{t = 2, u = 11} \Rightarrow$ INTERSECT AT $(16, 4)$

$\vec{r}_1{}'(t) = \langle 2t+6, -3t^2 \rangle \quad \vec{r}_1{}'(2) = \langle 10, -12 \rangle$

$\vec{r}_2{}'(u) = \langle 2, 0 \rangle \quad \vec{r}_2{}'(11) = \langle 2, 0 \rangle$

DIRECTION VECTORS AT THE INTERSECTION

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$20 + 0 = \sqrt{10^2 + (-12)^2}\sqrt{2^2 + 0^2} \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{20}{\sqrt{244}\; 2}\right) = \cos^{-1}\left(\frac{10}{\sqrt{244}}\right) \approx 50.1944^\circ$$

$$\approx 50 \text{ degrees}$$

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4. ([redacted] points) You are sitting at the origin on the surface $4z - x^2 - y^2 = 0$. You launch a water balloon into the air and its position at time t seconds is given roughly by the vector function $\mathbf{r}(t) = \langle t, 2t, 20t - 5t^2 \rangle$.

2 (a) Give the two word name of this surface.

(ELLIPTICAL PARABOLOID, OKAY)

CIRCULAR PARABOLOID

3 (b) Your math instructor just happens to be sitting at the location where the water balloon lands on the surface. Find the (x, y, z) location where your math instructors is sitting.

INTERSECTION: $4(20t - 5t^2) - t^2 - (2t)^2 = 0$

$$80t - 20t^2 - t^2 - 4t^2 = 0$$

$$80t - 25t^2 = 0$$

$$5t(16 - 5t) = 0 \qquad t = 0 \text{ or } \boxed{t = \tfrac{16}{5} = 3.2}$$

$$t = 3.2 \Rightarrow \boxed{(x, y, z) = \left(\tfrac{16}{5}, \tfrac{32}{5}, \tfrac{64}{5}\right) = (3.2, 6.4, 12.8)}$$

4 (c) Find parametric equations for the tangent line to the path at $t = 2$.

$$\vec{r}(2) = \langle 2, 4, 40 - 20 \rangle = \langle 2, 4, 20 \rangle$$

$$\vec{r}\,'(t) = \langle 1, 2, 20 - 10t \rangle \qquad \vec{r}\,'(2) = \langle 1, 2, 0 \rangle$$

$$\boxed{\begin{aligned} x &= 2 + t \\ y &= 4 + 2t \\ z &= 20 \end{aligned}}$$

ASIDE: THIS IS THE HIGHEST THE BALLOON GETS. THE TANGENT IS PARALLEL TO THE XY-PLANE.

4 (d) Find the curvature at time $t = 2$.

$$\vec{r}\,'(2) = \langle 1, 2, 0 \rangle$$

$$\vec{r}\,''(t) = \langle 0, 0, -10 \rangle \Rightarrow \vec{r}\,''(2) = \langle 0, 0, -10 \rangle$$

$$\vec{r}\,'(2) \times \vec{r}\,''(2) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & 0 & -10 \end{vmatrix} = (-20 - 0)\vec{i} - (-10 - 0)\vec{j} + (0 - 0)\vec{k} = \langle -20, 10, 0 \rangle$$

$$K(2) = \frac{|\vec{r}\,'(2) \times \vec{r}\,''(2)|}{|\vec{r}\,'(2)|^3} = \frac{\sqrt{20^2 + 10^2 + 0^2}}{(1^2 + 2^2 + 0^2)^{3/2}} = \boxed{\frac{\sqrt{500}}{5^{3/2}}}$$

$$\boxed{= \frac{10\sqrt{5}}{5^{3/2}} = \frac{10}{5} = 2}$$

ASIDE: THIS IS THE MAXIMUM CURVATURE FOR THIS CURVE