Solutions to Math 126 D Autumn 2024 Midterm I

1. (a) Using $\vec{BD} = \langle -2, 4, -1 \rangle$ and point $B(5, -2, 2)$ the intersection point will be $(5 + \frac{-2}{2}, -2 + \frac{4}{2}, 2 + \frac{-1}{2}) = (4, 0, \frac{3}{2})$.

   (b) This is the angle between the vectors $\vec{AC}$ and $\vec{BD}$.

$$\vec{AC} = \vec{AD} + \vec{AB} = \langle 1, 3, -2 \rangle + \langle 3, -1, -1 \rangle = \langle 4, 2, -3 \rangle$$

   and

$$\vec{BD} = \langle -2, 4, -1 \rangle$$

   so

$$\cos\theta = \frac{\langle 4, 2, -3 \rangle \cdot \langle -2, 4, -1 \rangle}{|\langle 4, 2, -3 \rangle||\langle -2, 4, -1 \rangle|} = \frac{-8 + 8 + 3}{\sqrt{16 + 4 + 9}\sqrt{4 + 16 + 1}}$$

   which gives

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{609}}\right)$$

   (c) Using the cross product

$$\langle 3, -1, -1 \rangle \times \langle 1, 3, -2 \rangle = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & -1 \\ 1 & 3 & -2 \end{bmatrix} = \langle 5, 5, 10 \rangle$$

   so the area is $\sqrt{25 + 25 + 100} = 5\sqrt{6}$.

2. (a) The direction vector of the line is given by

$$\langle 2, -3, 0 \rangle \times \langle 1, 2, -1 \rangle = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 1 & 2 & -1 \end{bmatrix} = \langle 3, 2, 7 \rangle$$

   To get a point, let $x = 0$, then $y = -3$ from the first plane equation which gives $z = x + 2y + 4 = -6 + 4 = -2$ from the second equation. (There are infinitely possible points to be found here.) So a vector equation for the line of intersection is

$$\mathbf{r}(t) = \langle 3t, -3 + 2t, -2 + 7t \rangle .$$

   (b) The line will contain the point $(0, -3, -2)$, for example. Then,a vector from the line to $(3, 1, -4)$ is $\langle 3 - 0, 1 - (-3), -4 - (-2) \rangle = \langle 3, 4, -2 \rangle$ so a normal to the plane is

$$\mathbf{n} = \langle 3, 2, 7 \rangle \times \langle 3, 4, -2 \rangle = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 7 \\ 3 & 4 & -2 \end{bmatrix} = \langle -32, 27, 6 \rangle$$

   and the plane equation is

$$-32(x - 3) + 27(y - 1) + 6(z + 4) = 0$$

   or $-32x + 27y + 6z = -93$.

3. (a) The radius of the largest circle is $r = 17/\pi$ so the ellipsoid equation is

$$\left(\frac{x}{\frac{17}{\pi}}\right)^2 + \left(\frac{y}{\frac{17}{\pi}}\right)^2 + \left(\frac{z}{10}\right)^2 = 1.$$

(b) This has many answers: $z = x^2$ ($z = ax^2 + b$ with $a \neq 0$ and b any number) or $y = x^2$ ($y = ax^2 + b$ with $a \neq 0$ and b any number)

(c) This also has many answers. One is

$$x = 2\cos t, y = t, z = 3\sin t.$$

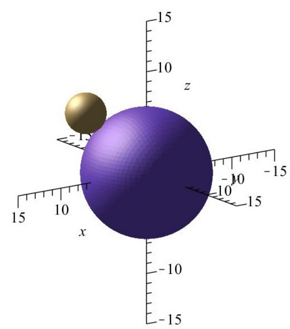
Others are given by

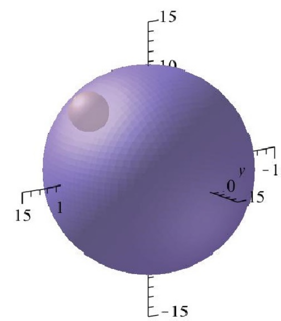
$$x = 2\cos(at), y = bt, z = 3\sin(at)$$

where a and b are non-zero numbers. You can also switch the sine and cosine functions.

(d) The distance of the center of the first sphere to the origin is $\sqrt{25+9+36} = \sqrt{70}$ and its radius is 2. So if the sphere centered at the origin is just going to touch the one centered at $(5, 3, 6)$

$$r = \sqrt{70} \pm 2$$





4. Given

$$\mathbf{r}(t) = \left\langle 2t+1, t-5, t^2 - t + 2 \right\rangle$$

(a) The point corresponds to $t = 3$. From

$$\mathbf{r}'(t) = \langle 2, 1, 2t-1 \rangle$$

we get the direction vector

$$\mathbf{r}'(3) = \langle 2, 1, 5 \rangle$$

so a vector equation for the line is

$$\mathbf{r}_L(t) = \langle 7+2t, -2+t, 8+5t \rangle$$

(b) From

$$\mathbf{r}''(t) = \langle 0, 0, 2 \rangle$$

we get the curvature

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\langle 2, 1, 2t-1 \rangle \times \langle 0, 0, 2 \rangle}{|\langle 2, 1, 2t-1 \rangle|^3} = \frac{|\langle 2, -4, 0 \rangle|}{(4+1+(2t-1)^2)^{3/2}} = \frac{\sqrt{20}}{(5+(2t-1)^2)^{3/2}}.$$

(c) The numerator of $\kappa(t)$ is constant. To maximize the quotient, we minimize the denominator so $2t - 1 = 0$ or $t = 1/2$.

$$\kappa(1/2) = \frac{\sqrt{20}}{5^{3/2}} = \frac{2}{5}$$

which happens at the position

$$\mathbf{r}(1/2) = \left\langle 2, -\frac{9}{2}, \frac{7}{4} \right\rangle$$