Autumn 2023 Su

MATH 126 Midterm Exam 1

Tuesday October 24, 50 minutes during your quiz section

Name: ______ Quiz section: ______

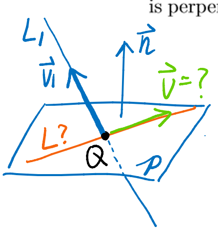
7-digit UW ID: ______

Exam Instruction:

- **You have 50 minutes to complete the exam. The exam contains 4 multi-part questions. Distribute your time accordingly.**
- Show your work to earn full credit. If you can not completely solve a problem, providing reasonable work and steps may still earn you some partial credit.
- The last page is a scratch paper, tear it off and do NOT turn it in unless you have written down additional work on it to be graded.
- **Do NOT write within 1 cm of the edge!** Your exam will be scanned for grading. If you run out of space, specify "see scratch paper", then write you additional work on the scratch paper and turn it in together with your exam.
- You can bring one hand-written double-sided 8.5" × 11" page of notes.
- The only allowed calculator is TI-30X IIS.
- Cell phone must be turned off and put away for the duration of the exam.
- **You must finish the exam independently. Giving or receiving any assistance on the exam is considered cheating, which will result in a grade of zero for the exam.**
- **Do NOT discuss the exam questions in person or online after your exam.**

1. (10 pts) Part (a), (b) are independent. Show your work to earn partial credit.

(a) Find parametric equations of a line that is contained in the plane $\overbrace{2x+y+3z=5}^{P}$, and is perpendicular to and intersects the line $\underbrace{x=1, y=2}_{L_1}$. The answer may not be unique.



- Find pt on L: since L intersects L_1 & L is on the plane P. the pt Q where L_1 intersects plane P is a pt on L.

Substitute L_1: $x=1, y=2$ into $2x+y+3z=5$

$\Rightarrow 2+2+3z=5 \Rightarrow z=\frac{1}{3}$

$\Rightarrow$ pt $Q(1,2,\frac{1}{3})$

- Find a vector $\vec{v}$ on L:

L_1: $x=1, y=2$ has vector $\vec{v_1}=\langle 0,0,1\rangle$.

plane P has normal vector $\vec{n}=\langle 2,1,3\rangle$

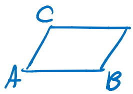
$$\left.\begin{array}{l} L \text{ in } P \Rightarrow \vec{v}\perp\vec{n} \\ L\perp L_1 \Rightarrow \vec{v}\perp\vec{v_1}\end{array}\right\} \Rightarrow \vec{v}=\vec{n}\times\vec{v_1}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 2 & 1 & 3\end{vmatrix}$$

$$=\langle -1,2,0\rangle$$

L: $x=1-1t$
$y=2+2t$
$z=\frac{1}{3}+0t$

(b) Find the area of the triangle with vertices $A(1,0,1), B(1,3,2), C(0,1,2)$.

Area of $\Delta ABC = \frac{1}{2}$ Area of



$= \frac{1}{2}|\vec{AB}\times\vec{AC}|$ $\leftarrow$ may use any two vectors formed by A, B, C.

$$= \frac{1}{2}\left\|\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 1 \\ -1 & 1 & 1\end{vmatrix}\right\|$$

$= \frac{1}{2}|\langle 2,-1,3\rangle|$

$= \frac{1}{2}\sqrt{4+1+9} = \frac{1}{2}\sqrt{14}$

2. (10 pts) Part (a), (b) are independent. The answer is not unique. Show your work.

(a) Give an example of two **non-parallel** vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\text{proj}_{\mathbf{v}}\mathbf{w} = \langle 2, 1, 2\rangle$ and $\text{comp}_{\mathbf{v}}\mathbf{w} = -3$.

$$\text{proj}_{\vec{v}}\vec{w} = (\text{comp}_{\vec{v}}\vec{w})\frac{\vec{v}}{|\vec{v}|}$$

or $-c\langle 2,1,2\rangle$ for any $c>0$

$$\langle 2,1,2\rangle = -3\frac{\vec{v}}{|\vec{v}|} \Rightarrow \frac{\vec{v}}{|\vec{v}|} = -\frac{1}{3}\langle 2,1,2\rangle \quad \text{e.g. } \vec{v} = -\langle 2,1,2\rangle$$

or $= \langle -2,-1,-2\rangle$

Then $\text{comp}_{\vec{v}}\vec{w} = \dfrac{\vec{w}\cdot\vec{v}}{|\vec{v}|} = -3$ Let $\vec{w} = \langle x, y, z\rangle$

$$\Leftrightarrow \frac{\langle x,y,z\rangle\cdot\langle -2,-1,-2\rangle}{|-\langle 2,1,2\rangle|} = \frac{-2x-y-2z}{3} = -3$$

$$\Leftrightarrow 2x+y+2z=9, \quad \text{e.g } x=0,\ y=9,\ z=0.$$

$\mathbf{v} =$ $-\langle 2,1,2\rangle$ $\mathbf{w} =$ $\langle 0,9,0\rangle$

$-c\langle 2,1,2\rangle$ $(c>0)$ any $\vec{w}=\langle x,y,z\rangle$ where $2x+y+2z=9$

(b) Give an example of two distinct planes that intersect along the line
L: $x = 1 + 2t,\ y = 1 - t,\ z = 3 + t.$

It suffices to find two planes both containing the line.

L: pt $P(1,1,3)$, vector $\vec{v} = \langle 2,-1,1\rangle$

plane P contains line $L \Leftrightarrow$ normal vector $\vec{n} \perp \vec{v}$ & P contains any pt of L

$$\vec{n}\cdot\vec{v}=0 \Leftrightarrow \langle a,b,c\rangle\cdot\langle 2,-1,1\rangle = 0$$
$$2a-b+c=0$$

e.g $\vec{n}_1 = \langle 1,2,0\rangle$
$\vec{n}_2 = \langle 1,0,-2\rangle$ use pt $P(1,1,3)$ for both planes.

(other approaches ok)

plane 1: $1(x-1)+2(y-1)+0(z-3)=0$

plane 2: $1(x-1)+0(y-1)-2(z-3)=0$

3. (11 pts) A space curve is given by the vector function $\mathbf{r}(t) = \langle 2t\sin t,\ t^2+2,\ t\cos t\rangle$.

(a) Find the equation of a quadric surface that contains the curve. Identify which type of quadric surface is it.

$$\left.\begin{array}{l} x(t) = 2t\sin t \Rightarrow \sin t = \frac{x}{2t} \\ y(t) = t^2+2 \\ z(t) = t\cos t \Rightarrow \cos t = \frac{z}{t} \end{array}\right\} \Rightarrow \sin^2 t + \cos^2 t = \left(\frac{x}{2t}\right)^2 + \left(\frac{z}{t}\right)^2 = 1$$

$$\frac{x^2}{4} + z^2 = t^2$$

$$\frac{x^2}{4} + z^2 = y-2$$

elliptic paraboloid

(b) What is the trace (cross section) curve of the surface your found in (a) on the plane $z = 1$? Circle your choice from the following:

(A) parabola (B) circle (C) ellipse (D) hyperbola (E) line(s)

$z=1$, $\frac{x^2}{4} + z^2 = y-2 \Rightarrow \frac{x^2}{4} = z-3$ parabola

(c) Find the angle of intersection between the curve $\mathbf{r}(t)$ and the line $x = y-2 = 3z$ at the intersection point $(0, 2, 0)$. Leave your answer as an exact expression.

$\vec{r}'(t) = \langle 2\sin t + 2t\cos t,\ 2t,\ \cos t - t\sin t\rangle$

pt $(0,2,0)$ $y(t) = t^2+2 = 2 \Rightarrow t=0$, $\vec{r}'(0) = \langle 0,0,1\rangle$

line $x = y-2 = 3z$ has vector: $\vec{v} = \langle 1, 1, \frac{1}{3}\rangle$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{r}'(0)\cdot\vec{v}}{|\vec{r}'(0)|\,|\vec{v}|}\right) = \cos^{-1}\left(\frac{\frac{1}{3}}{\sqrt{1+1+\frac{1}{9}}}\right)$$

$$\overset{\text{or}}{=} \cos^{-1}\left(\frac{\frac{1}{3}}{\sqrt{\frac{19}{9}}}\right)$$

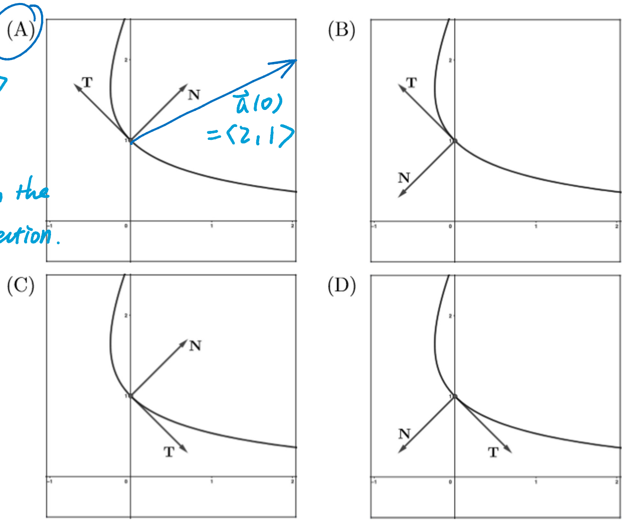
4. (11 pts) A particle moves along a curve in $\mathbb{R}^2$ with position vector function $\mathbf{r}(t) = \langle t^2 - t,\ e^t \rangle$.

(a) Which graph illustrates the unit tangent vector and the unit normal vector at $t = 0$? Circle your choice from (A)(B)(C)(D).

$\hat{T} = \frac{1}{\sqrt{2}}\langle -1, 1\rangle$

$\hat{N} \perp \hat{T}$

$\hat{N}$ point in the turning direction.



$\vec{r}'(t) = \langle 2t-1, e^t\rangle$

$\vec{r}'(0) = \langle -1, 1\rangle$

$\hat{T}(0) = \frac{\langle -1, 1\rangle}{|\langle -1, 1\rangle|} = \frac{1}{\sqrt{2}}\langle -1, 1\rangle$

(b) On the graph you chose above, sketch the acceleration vector $\mathbf{a}(t)$ at $t = 0$.

$\vec{a}(t) = \vec{r}''(t) = \langle 2, e^t\rangle \quad \vec{a}(0) = \langle 2, 1\rangle$

(c) Is the tangential component of the acceleration at $t = 0$ positive or negative? No need to explain your answer.

negative

$\vec{r}'(0) \cdot \vec{r}''(0) = \langle -1, 1\rangle \cdot \langle 2, 1\rangle = -1 < 0$

or see the angle between $\vec{v}(0)$ & $\vec{a}(0)$ is obtuse.

(d) Compute the curvature of the curve at $t = 0$.

$\vec{r}'(t) = \langle 2t-1, e^t\rangle \quad \vec{r}'(0) = \langle -1, 1\rangle$

$\vec{r}''(t) = \langle 2, e^t\rangle \quad \vec{r}''(0) = \langle 2, 1\rangle$

$$\kappa = \frac{|\vec{r}'(0) \times \vec{r}''(0)|}{|\vec{r}'(0)|^3} = \frac{|\langle -1, 1, 0\rangle \times \langle 2, 1, 0\rangle|}{|\langle -1, 1\rangle|^3} = \frac{|\langle 0, 0, -3\rangle|}{(\sqrt{2})^3} = \frac{3}{(\sqrt{2})^3}$$

(formula $\kappa(x) = \frac{|f''(x)|}{[1+f'(x)^2]^{3/2}}$ does not apply, $\vec{r}(t)$ can not be expressed as $y = f(x)$)