1. (12 pts) A particle moving along a curve has position vector $\mathbf{r}(t) = \langle t,\ t^2+1,\ t^2-t\rangle$.

(a) Find the time t at which the velocity vector is perpendicular to the acceleration vector.

velocity: $\vec{v}(t) = \vec{r}\,'(t) = \langle 1, 2t, 2t-1\rangle$

acceleration: $\vec{a}(t) = \vec{r}\,''(t) = \langle 0, 2, 2\rangle$

$\vec{v} \perp \vec{a} \iff \vec{v}(t)\cdot\vec{a}(t) = 0$

$\iff 0 + 4t + 2(2t-1) = 0$

$8t - 2 = 0 \Rightarrow \boxed{t = \frac{1}{4}}$

(b) Does the curve intersect the line $x = t+1$, $y = 2t+3$, $z = t$? (must show work to receive credit.)

curve: $\vec{r}_1(t)$, line: $\vec{r}_2(s)$, solve t & s from $\vec{r}_1(t) = \vec{r}_2(s)$

$$\begin{cases} ① & t = s+1 \\ ② & t^2+1 = 2s+3 \\ ③ & t^2 - t = s \end{cases}$$

plug in

$\Rightarrow (s+1)^2+1 = 2s+3$

$s^2+2s+2 = 2s+3 \Rightarrow s^2 = 1 \Rightarrow \begin{cases} s = +1 \\ t = 2\end{cases}$ or $\begin{cases} s=-1 \\ t = 0\end{cases}$

$\overset{①}{\Rightarrow}$

check ③: $t^2 - t \overset{?}{=} s$: $2^2 - 2 \neq 1$ $\quad 0^2 - 0 \neq -1$

$\Rightarrow$ no solution, $\boxed{\text{not intersecting}}$

[or observe that for $\vec{r}_1(t)$, $y - z = (t^2+1) - (t^2 - t) = t+1 = x+1$

but for $\vec{r}_2(s)$, $y - z = 2s+3-s = s+3 = x+2$

$\Rightarrow$ no intersection pt.]

2. (12 pts) Let C be the curve of intersection of

$$\text{the plane } \ x - 2y - z = 1 \quad \text{and the cylinder } \ y^2 + z^2 = 4$$

(a) Find a vector function $\mathbf{r}(t)$ for the curve C. Do not involve any square root function.

$y^2+z^2=4 \rightsquigarrow$ let $y = 2\cos t$

$z = 2\sin t$

$x-2y-z=1 \rightsquigarrow$ let $x = 2y+z+1 = 4\cos t + 2\sin t + 1$

$$\boxed{\vec{r}(t) = \langle 4\cos t + 2\sin t + 1, \ 2\cos t, \ 2\sin t \rangle}$$

(b) Find a line that intersects the curve C at the point $(3, 0, 2)$ and is perpendicular to the curve at this point (i.e., the angle of intersection between the curve and the line at the point is $\pi/2$). The answer is not unique, give a parametric equation for one such line that is not parallel to the x, y, or z–axis.

For $\vec{r}(t)$, the pt $(3, 0, 2)$ is at $t = \frac{\pi}{2}$

$\vec{r}'(t) = \langle -4\sin t + 2\cos t, \ -2\sin t, \ 2\cos t \rangle$

$\vec{r}'(\frac{\pi}{2}) = \langle -4, -2, 0 \rangle$

The line is $\perp$ the curve @ $(3, 0, 2)$ $\Rightarrow$ The line is $\perp$ the tangent vector $\vec{r}'(\frac{\pi}{2})$

For vector $\vec{v} = \langle a, b, c \rangle$ along the line, $\vec{v} \perp \vec{r}'(\frac{\pi}{2})$

$\Leftrightarrow \langle a, b, c \rangle \cdot \langle -4, -2, 0 \rangle = 0$

$\Leftrightarrow -4a - 2b = 0$ for example: let $a = 1$, $b = -2$, $c = 1$

To make sure the line is not parallel to the coordinate axis we want at least two of a, b, c be nonzero,

$\Rightarrow$ One such line is
$$\boxed{\begin{aligned} x &= 3 + 1t \\ y &= 0 - 2t \\ z &= 2 + 1t \end{aligned}}$$

answer not unique

3. (7 pts) Find an equation for the surface containing all the points whose distance to the point $(0,3,0)$ is twice its distance to the xz–plane. Simplify the equation and identify it as one of the six standard forms of quadric surfaces.

$$\sqrt{(x-0)^2+(y-3)^2+(z-0)^2} = 2|y|$$

$$x^2+(y-3)^2+z^2 = 4y^2$$

$$x^2+z^2 = 4y^2-y^2+6y-9$$

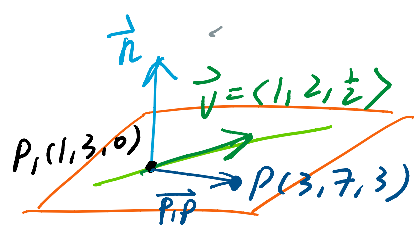
$$x^2+z^2 = 3y^2+6y-9$$

$$x^2+z^2 = 3(y^2+2y+1-1)-9$$

$$x^2+z^2 = 3(y+1)^2-12$$

$\boxed{-x^2+3(y+1)^2-z^2=12 \Rightarrow \text{hyperboloid of two sheets}}$

4. (7 pts) Find an equation for the plane that contains the point $P(3,7,3)$ and the line $x-1=\dfrac{y-3}{2}=2z$.



The line $\rightsquigarrow$ $\begin{cases} \text{pt } P_1(1,3,0) \\ \text{vector } \vec{v}=\langle 1,2,\frac{1}{2}\rangle \end{cases}$

normal vector of the plane $\vec{n}=\vec{v}\times\overrightarrow{P_1P}$

$$=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1/2 \\ 2 & 4 & 3 \end{vmatrix} = \langle 4,-2,0\rangle$$

$\Rightarrow$ plane: $\boxed{4(x-3)-2(y-7)+0(z-3)=0}$

5. (12 pts) Given that $\mathbf{a} = \langle 2, 1, 2\rangle$ and $\text{comp}_{\mathbf{a}}\, \mathbf{b} = 5$ (i.e., the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is 5).

(a) Find $\text{proj}_{\mathbf{a}}\, \mathbf{b}$.

$$\text{proj}_{\vec{a}} \vec{b} = (\text{comp}_{\vec{a}} \vec{b})\left(\frac{\vec{a}}{|\vec{a}|}\right) = 5\, \frac{\langle 2,1,2\rangle}{|\langle 2,1,2\rangle|} = \boxed{\frac{5}{3}\langle 2,1,2\rangle}$$

(b) Find $\mathbf{a} \cdot \mathbf{b}$.

$$\left.\begin{array}{l} \text{comp}_{\vec{a}} \vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = 5 \\ |\vec{a}| = |\langle 2,1,2\rangle| = 3 \end{array}\right\} \Rightarrow \vec{a}\cdot\vec{b} = 5\times 3 = \boxed{15}$$

(c) Find an example of such a vector $\mathbf{b}$ that is not parallel to $\mathbf{a}$. The answer is not unique.

Let $\vec{b} = \langle x, y, z\rangle$, $\vec{a}\cdot\vec{b} = \langle 2,1,2\rangle \cdot \langle x, y, z\rangle \overset{(b)}{=} 15$

$$\Rightarrow \boxed{2x + y + 2z = 15}$$

For example, if we let $y=0$, $z=0$, $x = \frac{15}{2}$

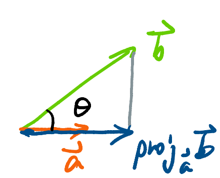
$$\Rightarrow \vec{b} = \boxed{\langle \tfrac{15}{2}, 0, 0\rangle}$$ answer not unique

(d) If you are also given that $|\mathbf{a} \times \mathbf{b}| = 12$, find the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{cases} |\vec{a}\times\vec{b}| = 12 \Rightarrow |\vec{a}||\vec{b}|\sin\theta = 12 & ① \\ \vec{a}\cdot\vec{b} = 15 \Rightarrow |\vec{a}||\vec{b}|\cos\theta = 15 & ② \end{cases}$$

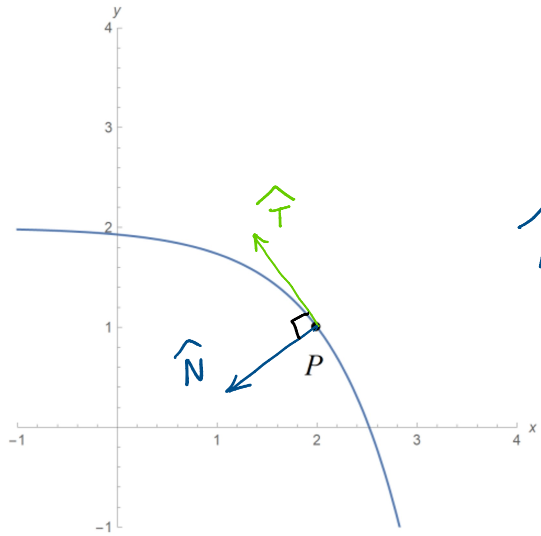
$$\frac{①}{②}: \tan\theta = \frac{12}{15} \Rightarrow \tan\theta = \frac{4}{5} \Rightarrow \theta = \boxed{\tan^{-1}\left(\frac{4}{5}\right)}$$

(since $\text{comp}_{\vec{a}}\vec{b} > 0$, θ is acute)



6. (12 pts) A particle is moving along the curve in the graph below. At the point P, the velocity vector of the particle is $\mathbf{v} = \langle -3, 4 \rangle$ and it is speeding up at a rate of 15 m/sec^2. You are also given that the curvature of the curve at the point P is $\kappa = \frac{2}{5}$.

(a) On the given graph, sketch the unit tangent vector $\mathbf{T}$ and the unit normal vector $\mathbf{N}$ at the point P. Then explicitly find the vectors $\mathbf{T}$ and $\mathbf{N}$ at the point P.



$\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -3, 4\rangle}{|\langle -3, 4\rangle|} = \boxed{\frac{1}{5}\langle -3, 4\rangle}$

$\hat{N}$ is pointing towards the turning direction

$\Rightarrow \left.\begin{array}{l} \hat{N} = \langle -, - \rangle \\ \hat{N} \perp \hat{T} \end{array}\right\} \Rightarrow \boxed{\hat{N} = \frac{1}{5}\langle -4, -3\rangle}$

(b) Find the tangential component a_T and the normal component a_N of the acceleration vector at the point P.

a_T = rate of change of speed $= \boxed{+15}$ m/s^2

$a_N = (\text{curvature})(\text{speed})^2 = \frac{2}{5} \cdot |\vec{v}|^2 = \frac{2}{5}(5)^2 = \boxed{10}$

(c) Use your answer in the previous parts to find the y–**component of the acceleration vector** at the point P.

$\vec{a} = a_T \hat{T} + a_N \hat{N} = 15 \frac{1}{5}\langle -3, 4\rangle + 10 \frac{1}{5}\langle -4, -3\rangle$

$= \langle -9 - 8, 12 - 6\rangle = \langle -17, 6\rangle$

$\Rightarrow$ y-component of $\vec{a}$ is $\boxed{6}$