1. (14 pts)

(a) Consider the line, L, that goes through the point $(0, 0, 0)$ and is orthogonal to the plane $2x - 3y + z = 5$. Find all (x, y, z) points of intersection of the line L and the surface $x^2 - \frac{7}{2}x + y^2 = 11z^2 + 4$.

L: $x = 0 + 2t,\ y = 0 - 3t,\ z = 0 + t$

INTERSECT: $(2t)^2 - \frac{7}{2}(2t) + (-3t)^2 = 11t^2 + 4$

$$4t^2 - 7t + 9t^2 = 11t^2 + 4$$

$$2t^2 - 7t - 4 = 0$$

$$(2t+1)(t-4) = 0$$

$$t = -\tfrac{1}{2} \quad \text{or} \quad t = 4$$

$$\boxed{(x,y,z) = (-1, \tfrac{3}{2}, -\tfrac{1}{2}) \quad \text{or} \quad (x,y,z) = (8, -12, 4)}$$

(b) Consider the line, L_1, that is given by the parametric equations $x = 13 + 2t$, $y = 5 - t$, $z = 3t$ and the line, L_2, that goes through the points $(0, 1, 0)$ and $(5, 2, 1)$. Find the (x, y, z) point of intersection of these lines, or show why the lines don't intersect.

L_2: $x = 0 + 5s,\ y = 1 + s,\ z = 0 + s$ ← DIRECTION: $\vec{v} = \langle 5,2,1\rangle - \langle 0,1,0\rangle$

INTERSECTION:
1. $13 + 2t = 5s$
2. $5 - t = 1 + s$
3. $3t = s$

→ 2 & 3 ⇒ $5 - t = 1 + 3t$

$4 = 4t$

$t = 1 \Rightarrow s = 3$

Check 1: $13 + 2(1) = 15$ ✓

$5(3) = 15$ ✓

Hence, $t = 1$, $s = 3$

So $\boxed{(x,y,z) = (15, 4, 3)}$

2. (12 points) For all parts below, consider the three points $A(2,0,2)$, $B(2,5,1)$ and $C(3,-2,5)$.

(a) Find the angle $\angle BAC$. (Give your answer rounded to the nearest degree)

$\vec{AB} = \langle 0,5,-1\rangle \quad \vec{AC} = \langle 1,-2,3\rangle$

$\vec{AB}\cdot\vec{AC} = |\vec{AB}||\vec{AC}|\cos\theta$

$0-10-3 = \sqrt{0+25+1}\sqrt{1+4+9}\cos\theta$

$\cos\theta = \dfrac{-13}{\sqrt{26}\sqrt{14}}$

$\theta = \cos^{-1}\left(\dfrac{-13}{\sqrt{26}\sqrt{14}}\right) \approx 132.95196^\circ \approx \boxed{133^\circ}$

(b) Find the area of the triangle ABC.

$\frac{1}{2}|\vec{AB}\times\vec{AC}|$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 5 & -1 \\ 1 & -2 & 3 \end{vmatrix} = (15-2)\vec{i}-(0--1)\vec{j}+(0-5)\vec{k} = \langle 13,-1,-5\rangle$$

$\text{AREA} = \frac{1}{2}\sqrt{13^2+(-1)^2+(-5)^2} = \frac{1}{2}\sqrt{169+1+25}$

$= \boxed{\frac{1}{2}\sqrt{195}} \quad \approx 6.9821$

(c) Find the equation of the plane containing A, B, and C.

$$\boxed{\begin{aligned} &13(x-2)-(y-0)-5(z-2)=0 \\ &13x-26-y-5z+10=0 \\ &13x-y-5z-16=0 \end{aligned}}$$

or any nonzero multiple

3. (14 pts)

(a) Consider the 'pretzel looking' parametric curve given by the equations

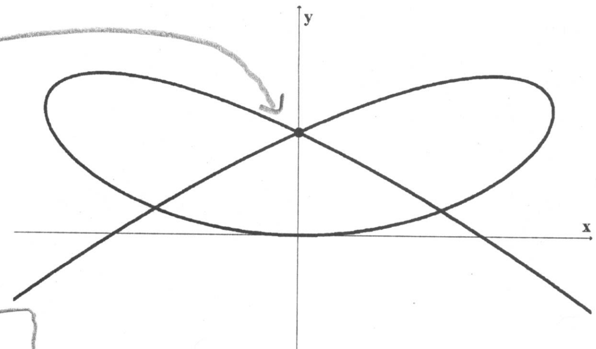
$$x = t^3 - 4t, \; y = 5t^2 - t^4.$$

The curve intersects the *positive* y-axis at the same y-intercept twice. Find the two different tangent slopes at this point.

$x = 0 \Rightarrow \quad 0 = t^3 - 4t$

$0 = t(t^2 - 4)$

$t = 0$ or $t = -2$ or $t = 2$



$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{10t - 4t^3}{3t^2 - 4}$$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{20 - 32}{12 - 4} = \frac{-12}{8} = -3/2$$

$$\left.\frac{dy}{dx}\right|_{t=-2} = \frac{-20 + 32}{12 - 4} = \frac{12}{8} = 3/2$$

(b) Consider the polar curve given by the equation $r = 5 - \sin(\theta^2 - 2\theta)$. ~~The curve has only one positive x-intercept~~. Find the equation for the tangent line at this positive x-intercept. (Your answer should be in terms of x and y).

$\theta = 0 \Rightarrow \quad r = 5 - \sin(0) = 5$

$$\frac{dr}{d\theta} = -(2\theta - 2)\cos(\theta^2 - 2\theta) \overset{\text{at } \theta = 0}{=} -(-2)\cos 0 = 2$$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \overset{\text{at } \theta = 0}{=} \frac{(2)(0) + (5)(1)}{(2)(1) - (5)(0)} = \frac{5}{2}$$

$x = 5, y = 0$

$$\boxed{y = \frac{5}{2}(x - 5) + 0 = \frac{5}{2}x - \frac{25}{2}}$$

4. (10 points) The motion of a particular fly in three-dimensions is described by the vector position function $\mathbf{r}(t) = \langle \cos(\pi t), t^3 - 1, \sin(\pi t) \rangle$.

(a) Eliminate the parameter and give the name of the surface of motion.

$y = t^3 - 1 \Rightarrow t = (y+1)^{1/3}$

$x^2 + z^2 = \cos^2(\pi (y+1)^{1/3}) + \sin^2(\pi (y+1)^{1/3}) = 1$

$x^2 + z^2 = 1$

$\boxed{\text{Circular cylinder}}$ around the y-axis

(b) Find the equation for the tangent line to the curve at $t = \frac{1}{2}$. And determine the (x, y, z) point of intersection of this tangent line with the xz-plane.

$\vec{r}(\tfrac{1}{2}) = \langle 0, \tfrac{1}{8} - 1, 1 \rangle = \langle 0, -\tfrac{7}{8}, 1 \rangle$

$\vec{r}\,'(t) = \langle -\pi \sin(\pi t), 3t^2, \pi \cos(\pi t) \rangle$

$\vec{r}\,'(\tfrac{1}{2}) = \langle -\pi, \tfrac{3}{4}, 0 \rangle$

TANGENT LINE :

$$\boxed{\begin{aligned} x &= -\pi t \\ y &= -\tfrac{7}{8} + \tfrac{3}{4} t \\ z &= 1 \end{aligned}}$$

xz-plane $\Rightarrow y = 0 \Rightarrow 0 = -\frac{7}{8} + \frac{3}{4} t$

$\frac{7}{8} = \frac{3}{4} t$

$t = \frac{7}{8} \cdot \frac{4}{3} = \frac{7}{6}$

$$\boxed{(x, y, z) = \left(-\tfrac{7\pi}{6}, 0, 1\right)}$$