1. (14 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\displaystyle\int \frac{1}{\sqrt{x^2+2x+10}}\,dx = \int \frac{1}{\sqrt{(x+1)^2+9}}\,dx$

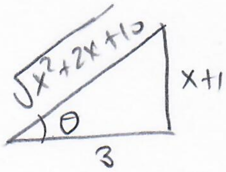
$\boxed{\begin{aligned} x+1 &= 3\tan\theta \\ dx &= 3\sec^2\theta\, d\theta \end{aligned}}$

$$= \int \frac{1}{\sqrt{9\tan^2\theta+9}}\, 3\sec^2\theta\, d\theta$$

$$= \int \frac{1}{\cancel{3}\,\cancel{\sec}\theta}\, \cancel{3}\sec^{\cancel{2}}\theta\, d\theta$$

$$= \int \sec\theta\, d\theta = \ln|\tan\theta + \sec\theta| + C$$

$\tan\theta = \frac{x+1}{3}$



$\therefore \sec\theta = \frac{\sqrt{x^2+2x+10}}{3}$

$$= \boxed{\ln\left|\frac{x+1}{3} + \frac{\sqrt{x^2+2x+10}}{3}\right| + C}$$

$$= \boxed{\ln\left(x+1+\sqrt{x^2+2x+10}\right) + C}$$

(b) $\displaystyle\int \sqrt{x}\, e^{\sqrt{x}}\, dx$ ① u-sub $\boxed{u^2 = x \quad 2u\,du = dx}$

$$= \int u e^u (2u\,du)$$

$$= \int 2u^2 e^u\, du$$

② IBP: $\left[\begin{array}{ll} w = 2u^2 & dv = e^u du \\ dw = 4u\,du & v = e^u \end{array}\right.$

$$= 2u^2 e^u - \int 4u e^u\, du$$

③ IBP $\left[\begin{array}{ll} w = 4u & dv = e^u du \\ dw = 4\,du & v = e^u \end{array}\right.$

$$= 2u^2 e^u - \left[4ue^u - \int 4e^u\, du\right]$$

$$= 2u^2 e^u - 4ue^u + 4e^u + C$$

$$= \boxed{e^{\sqrt{x}}\left(2x - 4\sqrt{x} + 4\right) + C}$$

2. (14 points) Evaluate the following integrals. Show all work. Simplify and box your answers.

(a) (5 points) $\int_0^{\frac{\pi}{3}} \frac{\sin(4x)}{\cos(2x)} dx$

$$= \int_0^{\pi/3} \frac{2\sin(2x)\cancel{\cos(2x)}}{\cancel{\cos(2x)}} dx$$

$$= \int_0^{\pi/3} 2\sin(2x)\, dx$$

$$= -\frac{\cancel{2}\cos(2x)}{\cancel{2}}\Big|_0^{\pi/3}$$

$$= -\cos\left(\frac{2\pi}{3}\right) + \cos(0)$$

$$= -(-1/2) + 1 = \boxed{3/2}$$

(b) (9 points) $\int_1^{\infty} \frac{x-2}{x^3+x} dx$

① Partial Fractions

$$\frac{x-2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + x(Bx+C)}{x(x^2+1)}$$

$$x - 2 = (A+B)x^2 + Cx + A$$

$$\begin{cases} A+B=0 \\ C=1 \\ A=-2 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=2 \\ C=1 \end{cases}$$

$$\int \frac{x-2}{x^3+x} dx = \int \frac{-2}{x} + \frac{2x+1}{x^2+1} dx$$

$$= -2\int \frac{1}{x} dx + \int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx$$

$$= \boxed{-2\ln|x| + \ln|x^2+1| + \arctan(x) + C}$$

$$= \ln\left(\frac{x^2+1}{x^2}\right) + \arctan(x) + C$$

$$= \boxed{\ln\left(1+\frac{1}{x^2}\right) + \arctan x + C}$$

$$\therefore \int_1^{\infty} \frac{x-2}{x^3+x} dx = \lim_{b\to\infty} \left[\ln\left(1+\frac{1}{x^2}\right) + \arctan x\right]\Big|_1^b$$

$$= \lim_{b\to\infty}\left[\ln\left(1+\frac{1}{b^2}\right) + \arctan b\right] - \ln(2) - \overbrace{\arctan(1)}^{=\pi/4}$$

$$= 0 + \underbrace{\lim_{b\to\infty} \arctan b}_{=\pi/2} - \ln(2) - \frac{\pi}{4} = \boxed{\frac{\pi}{4} - \ln 2}$$

3. (12 points) A train is traveling back and forth along a railway track, starting 4 miles south of King Street Station, heading north. We measure its velocity at t hours to be $v(t) = 60\sin(2\pi t)$ miles per hour.

(a) Where is the train in relation to King Street Station after 45 minutes?

$X(t)$ = position of train relative to King Street Station (positive = North)

Given: $X(0) = -4$ miles, $X'(t) = v(t) = 60\sin(2\pi t)$ mph.

Want to know: $X(3/4) = ?$ (45 min = 3/4 hrs)

$$X(t) = \int 60\sin(2\pi t)\,dt = 60\,\frac{-\cos(2\pi t)}{2\pi} + C = -\frac{30}{\pi}\cos(2\pi t) + C$$

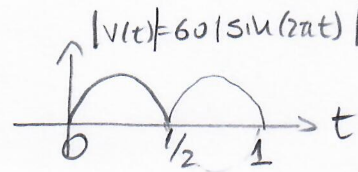
$$X(0) = -4: \quad -\frac{30}{\pi} + C = -4 \Rightarrow C = \frac{30}{\pi} - 4$$

$$\therefore X(t) = -\frac{30}{\pi}\cos(2\pi t) + \frac{30}{\pi} - 4$$

$$X(3/4) = \frac{-30}{\pi}\cos\left(\frac{3\pi}{2}\right) + \frac{30}{\pi} - 4 = \frac{30}{\pi} - 4 \cong 5.5493$$

Train is $\boxed{\frac{30}{\pi} - 4 \cong 5.5493 \text{ miles north of King Street Station}}$

(b) How many miles did the train travel in one hour?



total distance in 1 hour

$$= \int_0^1 |60\sin(2\pi t)|\,dt$$

$$= 2\int_0^{1/2} 60\sin(2\pi t)\,dt$$

$$= 2\left(\frac{-60\cos(2\pi t)}{2\pi}\Big|_0^{1/2}\right)$$

$$= \frac{60}{\pi}\left[-\cos(\pi) + \cos(0)\right]$$

$$= \frac{60}{\pi}\left[-(-1) + 1\right]$$

$$= \boxed{\frac{120}{\pi} \cong 38.1972 \text{ miles}}$$

4. (10 points) Set up an integral for the length of the curve $y = 4 - x^2$ from $x = -2$ to $x = 2$, and then approximate that integral using the Trapezoidal Rule with $n = 4$ subintervals. You may leave your answer in exact form, or as a decimal rounded to the nearest 2 decimal places.

$$L = \int_{-2}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \frac{dy}{dx} = -2x$$

$$= \int_{-2}^{2} \sqrt{1 + 4x^2}\, dx \qquad f(x) = \sqrt{1+4x^2}$$

$$\cong T_4 = \frac{\Delta x}{2}\left[f(-2) + 2f(-1) + 2f(0) + 2f(1) + f(2)\right]$$

$$= \frac{1}{2}\left[\sqrt{1+4(-2)^2} + 2\sqrt{1+4(-1)^2} + 2\sqrt{1+4(0)^2} + 2\sqrt{1+4(1)^2} + \sqrt{1+2(2)^2}\right]$$

$$= \frac{1}{2}\left[\sqrt{17} + 2\sqrt{5} + 2\sqrt{1} + 2\sqrt{5} + \sqrt{17}\right]$$

$$= \boxed{\sqrt{17} + 2\sqrt{5} + 1}$$

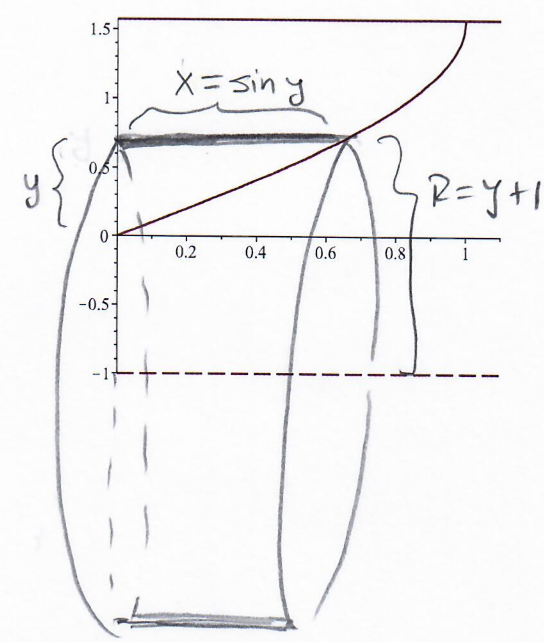
$$\cong \boxed{9.6}$$

5. (10 points) Let $\mathcal{R}$ be the region in the first quadrant bounded by:

$$\text{the lines } x = 0,\ y = \tfrac{\pi}{2} \text{ and the curve } y = \sin^{-1} x.$$

Rotate $\mathcal{R}$ about the line $y = -1$ to form a solid of revolution. What is the volume of this solid?

Best method: Shells (integrate in y)



$$V = \int_0^{\pi/2} 2\pi\,(\text{radius})(\text{height})\,dy$$

$$= \boxed{\int_0^{\pi/2} 2\pi\,(y+1)\sin y\,dy}$$

IBP: $u = (y+1)2\pi \quad dv = \sin y\,dy$

$du = 2\pi\,dy \quad v = -\cos y$

$$\therefore V = -2\pi(y+1)\cos y\,\Big|_0^{\pi/2} + 2\pi\int_0^{\pi/2}\cos y\,dy$$

$$= \left(-2\pi(y+1)\cos y + 2\pi\sin y\right)\Big|_0^{\pi/2}$$

$$= (0 + 2\pi) - (-2\pi + 0) = \boxed{4\pi}$$

If Disks/Washers (integral in x) then:

$$V = \int_0^1 \pi\left(\tfrac{\pi}{2}+1\right)^2 - \pi\left(\sin^{-1}x + 1\right)^2 dx$$

This would be very unpleasant to compute.

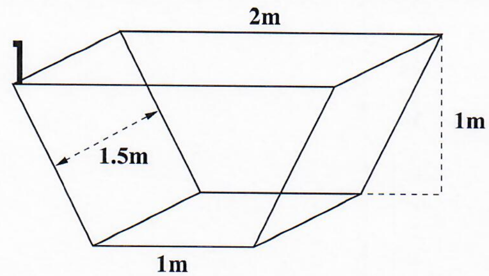
6. (10 points) A tank is completely filled with water. Recall that the density of water is 1000 kg/m^3 and the acceleration due to gravity is $g = 9.8$ m/sec^2.

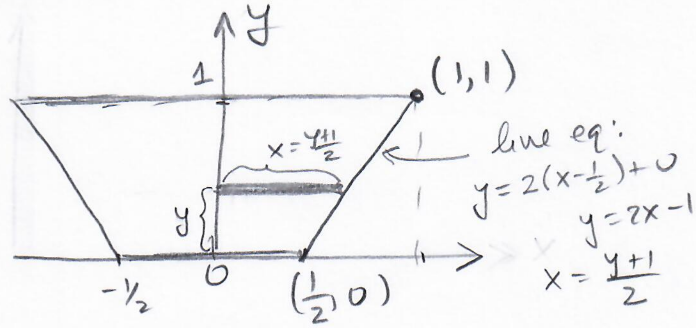
   The front and the back of the tank are 1.5 m apart and have the shape of a trapezoid. The trapezoids have width at bottom of 1 m, at top 2 m, and height 1 m (see picture.)

   The outlet of the water tank is 0.25 m higher than the top of the tank.

   Set up an integral for the work required to pump all the water out of the tank through the outlet.

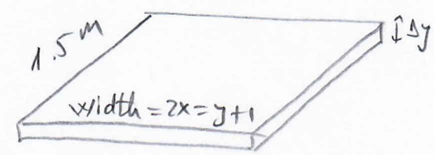
   SET UP ONLY. DO NOT EVALUATE THE INTEGRAL.





width across $= 2x = y+1$

A thin layer of water at height y above bottom of tank and of thickness Δy would be of rectangular shape, so it would have volume



$$\Delta V = (1.5\text{m})(\text{width})\,\Delta y$$
$$= (1.5)(y+1)\,\Delta y \quad (m^3)$$

and mass $1000\,\Delta V = 1500(y+1)\,\Delta y$ (kg)

so it would require a force of

$$\Delta F = (9.8)(1500)(y+1)\,\Delta y \quad \text{Newtons}$$

to move a distance $d = y_{outlet} - y = (1+0.25) - y = 1.25 - y$ meters

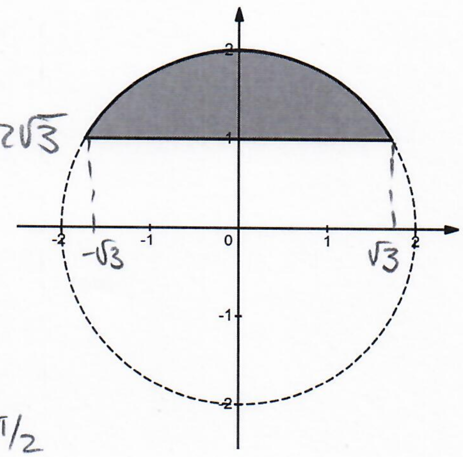
That is, the layer would require a work of

$$\Delta W = (\Delta F)(d) = (9.8)(1500)(y+1)(1.25-y)\,\Delta y \quad \text{Joules}$$

to pump to outlet. Adding up all layers, $0 \le y \le 1$, the total work is:

$$\boxed{W = \int_0^1 9.8(1500)(y+1)(1.25-y)\,dy}$$

7. (10 points) Find the y-coordinate $\bar{y}$ of the centroid of the region above the line $y = 1$ and below the curve $y = \sqrt{4-x^2}$.



① Area: $A = 2$ (area in 1st quadrant)

in x: $A = 2\int_0^{\sqrt{3}} \left(\sqrt{4-x^2} - 1\right)dx = 2\int_0^{\sqrt{3}} \sqrt{4-x^2}\,dx - 2\sqrt{3}$

OR

in y: $A = 2\int_1^2 \sqrt{4-y^2}\,dy$

Going with the latter: $y = 2\sin\theta$
$dy = 2\cos\theta\,d\theta$

$$\therefore A = 2\int_{\arcsin\frac{1}{2}=\frac{\pi}{6}}^{\arcsin(1)=\pi/2} \sqrt{4-4\sin^2\theta}\;2\cos\theta\,d\theta = 8\int_{\pi/6}^{\pi/2} \cos^2\theta\,d\theta$$

$$= \overset{4}{\cancel{8}}\int_{\pi/6}^{\pi/2} \frac{1+\cos(2\theta)}{\cancel{2}}\,d\theta = 4\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{\pi/6}^{\pi/2}$$

$$= 4\left(\frac{\pi}{2} - \frac{\pi}{6} + \frac{1}{2}\left(\overset{0}{\cancel{\sin}\,\pi} - \overset{\sqrt{3}/2}{\sin\frac{\pi}{3}}\right)\right) = \boxed{\frac{4\pi}{3} - \sqrt{3} = A}$$

② $M_x = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{2}\left(\sqrt{4-x^2}\right)^2 - \frac{1}{2}(1)^2\,dx = \cancel{2}\int_0^{\sqrt{3}} \frac{1}{\cancel{2}}(4-x^2) - \frac{1}{\cancel{2}}\,dx$

$$= \int_0^{\sqrt{3}} 3 - x^2\,dx = \left(3x - \frac{1}{3}x^3\right)\Big|_0^{\sqrt{3}} = 3\sqrt{3} - \frac{1}{3}(\sqrt{3})^3 = 3\sqrt{3} - \sqrt{3} = \overset{M_x}{\boxed{2\sqrt{3}}}$$

$$\therefore \bar{y} = \frac{1}{A}M_x = \frac{1}{\frac{4\pi}{3} - \sqrt{3}}(2\sqrt{3}) = \boxed{\frac{6\sqrt{3}}{4\pi - 3\sqrt{3}} = \bar{y} \cong 1.41}$$

8. (10 points) Find the solution of the following initial value problem. Write your answer in explicit form, $y = f(x)$.

$$\frac{dy}{dx} = \frac{x}{y}\sqrt{(x^2+1)(y^2+1)}, \qquad y(0) = \frac{3}{4}$$

$$\frac{dy}{dx} = \left(x\sqrt{x^2+1}\right)\left(\frac{\sqrt{y^2+1}}{y}\right)$$

$$\underbrace{\int \frac{y}{\sqrt{y^2+1}}\,dy}_{\substack{u\text{-sub} \\ \text{w/ } u=y^2+1 \text{ or } u=\sqrt{y^2+1} \\ [\ldots]}} = \underbrace{\int x\sqrt{x^2+1}\,dx}_{\substack{v\text{-sub} \\ \text{with } v=\sqrt{x^2+1} \text{ or } v=x^2+1 \\ [\ldots]}}$$

$$\sqrt{y^2+1} = \frac{1}{3}(x^2+1)^{3/2} + C$$

$y(0)=\frac{3}{4}$: $\sqrt{\frac{9}{16}+1} = \frac{5}{4} = \frac{1}{3}(0+1)^{3/2} + C \Rightarrow C = \frac{5}{4} - \frac{1}{3} = \frac{11}{12}$

$$\therefore \sqrt{y^2+1} = \frac{1}{3}(x^2+1)^{3/2} + \frac{11}{12}$$

$$\therefore y^2 \cancel{+1} = \left(\frac{1}{3}(x^2+1)^{3/2} + \frac{11}{12}\right)^2 - 1$$

$$y = \pm\sqrt{\left(\frac{1}{3}(x^2+1)^{3/2} + \frac{11}{2}\right)^2 - 1}$$

Since $y(0) = \frac{3}{4} > 0 \Rightarrow$ pick "+" solution

$$\boxed{y = \sqrt{\left(\frac{1}{3}(x^2+1)^{3/2} + \frac{11}{2}\right)^2 - 1}}$$

9. (10 points) At time $t = 0$, a tank contains 100 gallons of pure gasoline.

A mixture whose volume is 30% ethanol and 70% gasoline is pumped into the tank at a rate of 2 gallons per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate.

Calculate the number of gallons of ethanol in the tank after 25 minutes. Round your answer to two decimal places.

Let $y(t)$ = gallons of ethanol in the tank after t minutes.

Then $y(0) = 0$ and

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}) = (0.3)(2\text{ gall/min}) - \left(\frac{y}{100}\right)(2\text{ gall/min})$$

$$\therefore \frac{dy}{dt} = 0.6 - \frac{2y}{100} = 0.6 - \frac{y}{50} = \frac{30-y}{50}$$

Separating the variables & integrating

$$\int \frac{1}{30-y}\,dy = \int \frac{1}{50}\,dt$$

$$-\ln|30-y| = \frac{1}{50}t + C$$

$$\ln|30-y| = -\frac{1}{50}t + C_1$$

$$30 - y = Ae^{-1/50\,t}$$

$$y = 30 - Ae^{-1/50\,t}$$

$$y(0) = 0 \Rightarrow \quad 0 = 30 - A \Rightarrow A = 30$$

$$\therefore \boxed{y(t) = 30 - 30e^{-1/50\,t}}$$

After $t = 25$ min:

$$y(25) = 30 - 30e^{-25/50} = 30 - 30e^{-1/2}$$

$$= \boxed{30 - \frac{30}{\sqrt{e}} \cong 11.8 \text{ gallons}} \text{ of ethanol.}$$