1. (12 points) Evaluate the following integrals.

(a) (6 points) $\int \frac{11x-12}{x^3-4x^2+4x}\,dx$

$$\frac{11x-12}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$\Rightarrow \quad 11x-12 = A(x-2)^2 + Bx(x-2) + Cx$$

$$11x-12 = (A+B)x^2 + (-4A-2B+C)x + 4A$$

$$4A = -12 \Rightarrow A = -3$$

$$A+B=0 \Rightarrow B = -A = 3$$

$$-4A-2B+C = 11 \Rightarrow 12-6+C = 11 \Rightarrow C = 5$$

$$\int \frac{11x-12}{x(x-2)^2}\,dx = \int \frac{-3}{x} + \frac{3}{x-2} + \frac{5}{(x-2)^2}\,dx$$

$$= \boxed{-3\ln|x| + 3\ln|x-2| - \frac{5}{x-2} + C}$$

$$= \boxed{3\ln\left|\frac{x-2}{x}\right| - \frac{5}{x-2} + C}$$

(b) (6 points) $\int \frac{\tan^2(t)\sec^2(t)}{1+\tan(t)}\,dt$

$u = 1+\tan t$

$du = \sec^2 t\,dt$

$$\int \frac{(u-1)^2}{u}\,du$$

$$= \int \frac{u^2-2u+1}{u}\,du$$

$$= \int u - 2 + \frac{1}{u}\,du$$

$$= \frac{1}{2}u^2 - 2u + \ln|u| + C$$

$$= \boxed{\frac{1}{2}(1+\tan t)^2 - 2(1+\tan t) + \ln|1+\tan t| + C}$$

OR

$u = \tan t$

$du = \sec^2 t\,dt$

$$\int \frac{u^2}{1+u}\,du$$

$$= \int u - 1 + \frac{1}{u+1}\,du$$

$$\begin{array}{r} u-1 \\ u+1\,\overline{)\,u^2} \\ -(u^2+u) \\ \hline -u \\ -(-u-1) \\ \hline 1. \end{array}$$

$$= \frac{1}{2}u^2 - u + \ln|u+1| + C$$

$$= \boxed{\frac{1}{2}\tan^2 t - \tan t + \ln|\tan t + 1| + C}$$

(there are other equivalent answers possible)

2. (12 points)

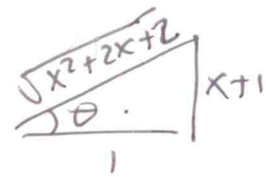
(a) (8 points) Evaluate the indefinite integral: $\displaystyle\int \frac{1}{(x^2+2x+2)^{3/2}}\,dx$

$$= \int \frac{1}{((x+1)^2+1)^{3/2}}\,dx \qquad \boxed{\begin{aligned} x+1 &= \tan\theta \\ dx &= \sec^2\theta\, d\theta \end{aligned}}$$

$$= \int \frac{1}{(\sec^2\theta)^{3/2}} \sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sec^3\theta} \sec^2\theta\, d\theta = \int \frac{1}{\sec\theta}\, d\theta$$

$$= \int \cos\theta\, d\theta = \sin\theta + C$$



$$= \boxed{\frac{x+1}{\sqrt{x^2+2x+2}} + C}$$

(b) (4 points) Use your answer from part (a) to determine if the following improper integral is convergent or divergent. If convergent, evaluate it. Show all limit computations.

$$\int_0^\infty \frac{1}{(x^2+2x+2)^{3/2}}\,dx$$

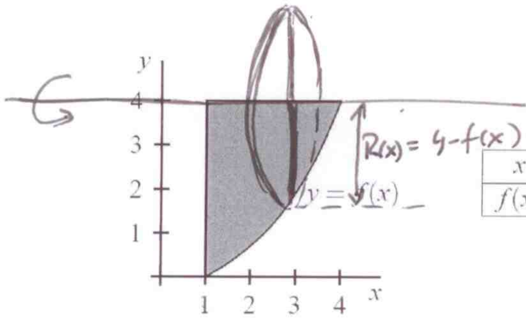
$$= \lim_{t\to\infty} \int_0^t \frac{1}{(x^2+2x+2)^{3/2}}\,dx = \lim_{t\to\infty}\left(\frac{t+1}{\sqrt{t^2+2t+2}} - \frac{0+1}{\sqrt{0^2+0+2}}\right)$$

(dividing by t top & bottom)

$$= \underbrace{\lim_{t\to\infty}\left(\frac{1+1/t}{\sqrt{1+2/t+2/t^2}}\right)} - \frac{1}{\sqrt{2}}$$

$$= 1 - \frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2}-1}{\sqrt{2}} = 1-\frac{1}{\sqrt{2}}}$$

3. (10 points) Consider the region between the curve $y = f(x)$, the line $x = 1$, and the line $y = 4$. A formula for $f(x)$ is not known; however, we do have the following picture of the region and a table of values:



x	1	1.5	2	2.5	3	3.5	4
$f(x)$	0	.3	.7	1.2	1.9	2.8	4

(a) (6 points) Use Simpson's Rule with $n = 6$ subintervals to estimate the volume of the solid of revolution obtained by rotating this region around the **horizontal** line $y = 4$.

First, setting up the volume integral, using Disks:

$$V = \int_1^4 \pi R^2(x)\,dx = \int_1^4 \pi\,(4 - f(x))^2\,dx \qquad \left(\Delta x = \frac{4-1}{6} = 0.5\right)$$

$$S_6 = \pi\,\frac{0.5}{3}\Big[(4-0)^2 + 4(4-0.3)^2 + 2(4-0.7)^2 + 4(4-1.2)^2 + 2(4-1.9)^2 + 4(4-2.8)^2 + (4-4)^2\Big]$$

$$= \frac{\pi}{6}\,[138.48] = \boxed{23.08\pi} \cong \boxed{72.50796} \text{ cubic units}$$

(b) (4 points) Define $A(x) = \int_1^{1+1/x} f(t)\,dt$, where f is the same function as in the table above. Compute $A'(2)$.

$$A'(x) = \frac{d}{dx}\int_1^{1+1/x} f(t)\,dt = f\left(1+\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)$$ by FTC I and Chain Rule

$$A'(2) = \underbrace{f\left(1+\frac{1}{2}\right)}\left(-\frac{1}{2^2}\right)$$

$$= 0.3\left(-\frac{1}{4}\right)$$

$$= \boxed{\frac{-0.3}{4}} = \boxed{-0.075}$$

4. (12 points) The velocity of a particle moving along the number line is given at all times $t \geq 1$ by

$$v(t) = \frac{8}{t^3} - \frac{4}{t^2} \text{ ft/sec.}$$

At time $t = 2$ seconds, the particle's position is $s(2) = 10$ feet.

(a) (6 points) Find the function, $s(t)$, for the position of the particle at time t seconds, $t \geq 1$.

$$S(t) = \int 8t^{-3} - 4t^{-2}\,dt = \frac{8}{-2}t^{-2} - \frac{4}{-1}t^{-1} + C$$

$$= -\frac{4}{t^2} + \frac{4}{t} + C$$

$$S(2) = 10 \Rightarrow -\frac{4}{(2)^2} + \frac{4}{(2)} + C = 10 \Rightarrow C = 9$$

$$\boxed{S(t) = -\frac{4}{t^2} + \frac{4}{t} + 9}$$

(b) (6 points) Find the **total distance** traveled by the particle from $t = 1$ to $t = 4$ seconds.

Total distance $= \int_1^4 |v(t)|\,dt = \int_1^4 \left|\frac{8}{t^3} - \frac{4}{t^2}\right| dt$

$$\frac{8}{t^3} - \frac{4}{t^2} = 0 \Leftrightarrow 8 - 4t = 0 \Leftrightarrow t = 2$$

$$\int_1^2 \frac{8}{t^3} - \frac{4}{t^2}\,dt = \left(-\frac{4}{t^2} + \frac{4}{t}\right)\Big|_1^2 = (-1+2) - (-4+4) = 1 \text{ ft}$$

$$\int_2^4 \frac{8}{t^3} - \frac{4}{t^2}\,dt = \left(-\frac{4}{t^2} + \frac{4}{t}\right)\Big|_2^4 = \left(-\frac{4}{16} + 1\right) - (-1+2) = -\frac{1}{4} \text{ ft}$$

Particle traveled 1 ft in the positive direction, then $\frac{1}{4}$ ft in the negative direction.

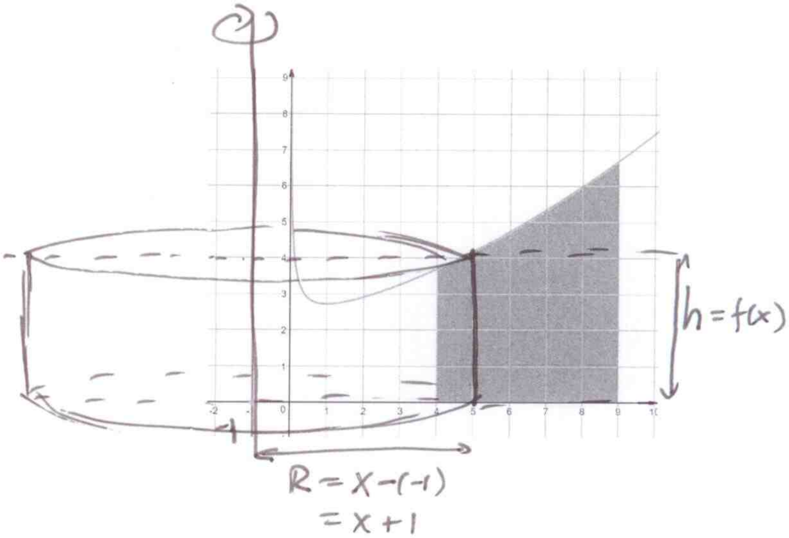
Total distance $= 1 + \frac{1}{4} = \boxed{\frac{5}{4} \text{ ft}}$.

5. (12 points) Find the volume of the body of revolution obtained by rotating the region below $y = \frac{e^{\sqrt{x}}}{\sqrt{x}}$ and above the x-axis, between $x = 4$ and $x = 9$, about the **vertical** line $x = -1$.

SHELLS:

$$V = \int_4^9 2\pi\, r(x)\, h(x)\, dx$$

$$= \int_4^9 2\pi (x+1) \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$$



$u = \sqrt{x} \Rightarrow u^2 = x$

$2u\,du = dx$

$$= \int_2^3 2\pi (u^2+1) \frac{e^u}{\cancel{u}}\, 2\cancel{u}\, du$$

$$= 4\pi \int_2^3 (u^2+1) e^u\, du$$

IBP: $w = u^2+1$ $\quad dv = e^u du$
$dw = 2u\,du$ $\quad v = e^u$

$$= 4\pi \left[(u^2+1)e^u \Big|_2^3 - \int_2^3 2ue^u\, du \right]$$

IBP: $w = 2u$ $\quad dv = e^u du$
$dw = 2du$ $\quad v = e^u$

$$= 4\pi \left[10e^3 - 5e^2 - 2ue^u\Big|_2^3 + \int_2^3 2e^u\, du \right]$$

$$= 4\pi \left[10e^3 - 5e^2 - 6e^3 + 4e^2 + 2e^u\Big|_2^3 \right]$$

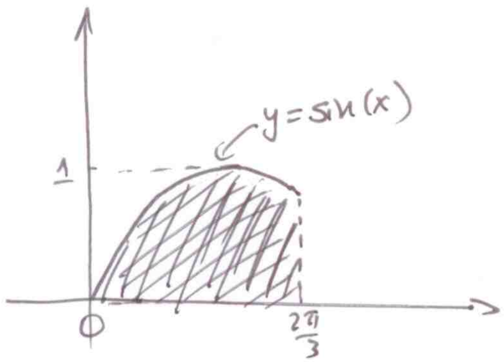
$$= 4\pi \left[4e^3 - e^2 + 2e^3 - 2e^2 \right]$$

$$= 4\pi \left[6e^3 - 3e^2 \right]$$

$$= \boxed{12\pi e^2 [2e-1]}$$

7. (12 points) Find the center of mass of the region under the graph of $y = \sin(x)$, above the x-axis, between $x = 0$ and $x = 2\pi/3$ radians.

   You may give your answer in exact, simplified form, or in decimal form, rounded to the nearest 4 decimal digits.



1) $A = \int_0^{2\pi/3} \sin(x)\,dx$

$= (-\cos x)\Big|_0^{2\pi/3}$

$= -\cos\frac{2\pi}{3} + \cos 0 = \frac{1}{2} + 1 = \frac{3}{2}$

2) The moment about the y-axis is:

$M_y = \int_0^{2\pi/3} x\sin x\,dx = x(-\cos x)\Big|_0^{2\pi/3} - \int_0^{2\pi/3} (-\cos x)\,dx$

$= (-x\cos x + \sin x)\Big|_0^{2\pi/3} = -\frac{2\pi}{3}\left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

3) $M_x = \int_0^{2\pi/3} \frac{1}{2}\sin^2 x\,dx = \int_0^{2\pi/3} \frac{1}{2}\,\frac{1-\cos 2x}{2}\,dx = \int_0^{2\pi/3} \frac{1}{4} - \frac{\cos 2x}{4}\,dx$

$= \left(\frac{x}{4} - \frac{1}{8}\sin(2x)\right)\Big|_0^{2\pi/3} = \left(\frac{\pi}{6} - \frac{1}{8}\sin\left(\frac{4\pi}{3}\right)\right) - (0)$

$= \frac{\pi}{6} - \frac{1}{8}\left(-\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} + \frac{\sqrt{3}}{16}$

4) $\bar{x} = \frac{M_y}{A} = \frac{\pi/3 + \frac{\sqrt{3}}{2}}{3/2} = \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)\cdot\frac{2}{3} = \frac{2\pi}{9} + \frac{\sqrt{3}}{3}$

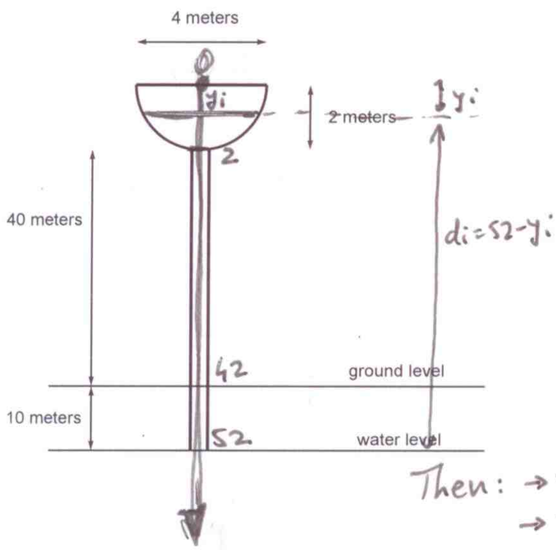
5) $\bar{y} = \frac{M_x}{A} = \frac{\pi/6 + \frac{\sqrt{3}}{16}}{3/2} = \left(\frac{\pi}{6} + \frac{\sqrt{3}}{16}\right)\cdot\frac{2}{3} = \frac{\pi}{9} + \frac{\sqrt{3}}{24}$

$$\boxed{(\bar{x}, \bar{y}) = \left(\frac{2\pi}{9} + \frac{\sqrt{3}}{3},\ \frac{\pi}{9} + \frac{\sqrt{3}}{24}\right) \cong (1.27548,\ 0.421235)}$$

6. (10 points) A water tower has a tank that has a hemispherical shape. Its dimensions are shown in the figure. The bottom of the tank is 40 meters above the ground. Water is pumped into the tower from a well 10 meters under ground.

   Compute the work required to completely fill the empty tank with water.

   Assume that water weighs 1000 kg/m^3, the gravitational acceleration is $g = 9.8$ m/s^2, and that the amount of water in the pipe between the water table and the tank is negligibly small.



Since we have to compute this integral, it's best to set up the coordinate system with origin at the top (center of circle) and positive direction downwards

Slice the interval $0 \le y \le 2$ into n subintervals each corresponding to a very thin slice of water, in the shape of a disk, of radius $R_i = \sqrt{4 - y_i^2}$ and thickness Δy

Then: → volume of i^{th} slice is: $V_i = \pi R_i^2 \Delta y = \pi(4 - y_i^2)\Delta y$ m^3

→ weight is: $F_i = 9800\pi(4 - y_i^2)\Delta y$, in Newtons

→ Slice was lifted a distance $d_i = 52 - y_i$ meters.

$$\text{Total work} = \lim_{n\to\infty} \sum_{i=1}^{n} 9800\pi(4 - y_i^2)(52 - y_i)\Delta y = \int_0^2 9800\pi(4 - y^2)(52 - y)\,dy$$

$$= \boxed{9800\pi \int_0^2 (4 - y^2)(52 - y)\,dy} = 9800\pi \int_0^2 208 - 4y - 52y^2 + y^3\,dy$$

$$= 9800\pi \left(208y - 2y^2 - \frac{52}{3}y^3 + \frac{1}{4}y^4\right)\Big|_0^2 = 9800\pi\,\frac{820}{3}$$

$$= \boxed{\frac{8{,}036{,}000\pi}{3} \text{ Joules}} \cong 8{,}415{,}279.52 \text{ J}$$

Other correct setups for the work integral:

w/ origin at bottom of sphere, pos. direction up:

$$\int_0^2 9800\pi\,(4 - (y-2)^2)(50 + y)\,dy$$

w/ origin at ground level (pos. dir. up): $\int_{40}^{42} 9800\pi(4 - (y-42)^2)(y+10)\,dy$

w/ origin at water level (pos. dir. up): $\int_{50}^{52} 9800\,(4 - (y-52)^2)(y)\,dy$

8. (10 points) Find the explicit solution $y = y(x)$ for the differential equation:

$$\frac{dy}{dx} = \left(\tan(x)\sec^2(x) - 4\right) e^{2y}$$

subject to the initial condition:

$$y(0) = 1.$$

Separating the variables and integrating:

$$\int e^{-2y}\,dy = \int \left(\tan(x)\sec^2(x) - 4\right) dx$$

$$-\frac{1}{2}e^{-2y} = \int \tan(x)\sec^2(x)\,dx - \int 4\,dx$$

$$-\frac{1}{2}e^{-2y} = \int u\,du - 4x$$

$$-\frac{1}{2}e^{-2y} = \frac{1}{2}u^2 - 4x + C$$

$$-\frac{1}{2}e^{-2y} = \frac{1}{2}\tan^2 x - 4x + C$$

Simplifying: $e^{-2y} = -\tan^2 x + 8x + C_1$ $\quad (C_1 = -2C)$

& $y(0)=1$: $e^{-2} = -\tan^2(0) + 8(0) + C_1 \Rightarrow C_1 = e^{-2}$

we get: $e^{-2y} = -\tan^2 x + 8x + e^{-2}$

Solving for y explicitly:

$$-2y = \ln\left(-\tan^2 x + 8x + e^{-2}\right)$$

$$\boxed{y = -\frac{1}{2}\ln\left(-\tan^2 x + 8x + e^{-2}\right)}$$

9. (10 points) At time $t = 0$, a tank contains 100 gallons of pure gasoline. A mixture whose volume is 30% ethanol and 70% gasoline is pumped into the tank at a rate of 2 gallons per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate.

   How many gallons of ethanol will there be in the tank after 50 minutes? Give your answer as a decimal number, rounded to the nearest 4 decimal digits.

Let $y(t)$ = gallons of ethanol in tank at t minutes

The rate at which ethanol comes into the tank is:

$$\text{rate in} = 30\% \text{ of } 2 \text{ gallons/min} = (0.3)(2) = 0.6 \text{ gall/min}.$$

The rate at which it's leaving the tank is:

$$\text{rate out} = \begin{pmatrix}\text{concentration}\\ \text{in tank}\end{pmatrix} \cdot \left(2 \text{ gall/min}\right) = \frac{y}{100} \cdot 2 = \frac{y}{50} \text{ gall/min}.$$

So, $y(t)$ satisfies the differential equation:

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}) = 0.6 - \frac{y}{50}$$

$$\frac{dy}{dt} = \frac{30-y}{50}$$

Separating the variables and integrating:

$$\int \frac{1}{30-y}\, dy = \int \frac{1}{50}\, dt$$

$$-\ln|30-y| = \frac{1}{50}t + C$$

$$\ln|30-y| = -\frac{1}{50}t + C_1 \qquad (C_1 = -C)$$

$$30 - y = A e^{-1/50 t} \qquad (A = \pm e^{C_1})$$

$$y = 30 - A e^{-1/50 t}$$

Fixing the constant: $y(0) = 0 \Rightarrow A = 30 \Rightarrow \boxed{y = 30 - 30 e^{-1/50 t}}$

At $t = 50$ minutes: $y = 30 - 30 e^{-1}$

$$\cong \boxed{18.9636 \text{ gallons}} \text{ of ethanol.}$$