1. (10 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\int x^2(1+x)^{2022}\,dx$

Solution:

(a) Substitution: $u = 1+x$, $du = dx$.

$$\int x^2(1+x)^{2022}\,dx = \int (u-1)^2u^{2022}du = \int (u^2-2u+1)u^{2022}du$$

$$= \int (u^{2024} - 2u^{2023} + u^{2022})du = \frac{1}{2025}u^{2025} - \frac{2}{2024}u^{2024} + \frac{1}{2023}u^{2023} + C$$

$$= \boxed{\frac{1}{2025}(1+x)^{2025} - \frac{2}{2024}(1+x)^{2024} + \frac{1}{2023}(1+x)^{2023} + C}$$

(b) $\int \frac{x}{\sqrt{x^2-6x+10}}\,dx$

$$x^2 - 6x + 10 = (x^2-6x+\overbrace{9) + 10 - 9}^{\text{cancel}} = (x-3)^2+1$$

$x-3 = \tan\theta$ $\Rightarrow$ $x = \tan\theta + 3$,

$dx = \sec^2\theta\,d\theta$ $\sqrt{(x-3)^2+1} = \sqrt{\tan^2\theta+1} = \sec\theta$

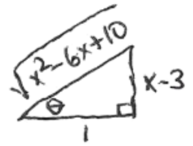
(or could do 2 substitutions, $u = x-3$ and $u = \tan\theta$)

$$\int \frac{x}{\sqrt{x^2-6x+10}}\,dx = \int \frac{\tan\theta + 3}{\sec\theta}\sec^2\theta\,d\theta$$

$$= \int \sec\theta\tan\theta + 3\sec\theta\,d\theta$$

$$= \sec\theta + 3\ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\sqrt{x^2-6x+10} + 3\ln\left|\sqrt{x^2-6x+10} + x - 3\right| + C}$$



$\tan\theta = x-3 = \frac{x-3}{1}$

$\sec\theta = \sqrt{x^2-6x+10}$

2. (10 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\int_0^{\sqrt{\pi}} x^3 e^{x^2}\, dx$

Subst. $w = x^2$ $\quad x = \sqrt{\pi} \Leftrightarrow w = \pi$

$dw = 2x\, dx$ $\quad x = 0 \Leftrightarrow w = 0$

$\frac{1}{2} dw = x\, dx$

$$\hookrightarrow = \int_0^{\sqrt{\pi}} x^2 \cdot e^{x^2} \cdot x\, dx = \frac{1}{2}\left[\int_0^{\pi} w e^w\, dw\right]$$

[5 points total]

int. by parts $\quad u = w \quad dv = e^w dw$

$du = dw \quad v = e^w$

$$= \frac{1}{2}\left[(we^w)\Big|_0^{\pi} - \int_0^{\pi} e^w\, dw\right]$$

$$= \frac{1}{2}\left[(we^w - e^w)\Big|_0^{\pi}\right]$$

$$= \boxed{\frac{1}{2}\left((\pi - 1)e^{\pi} + 1\right)}$$

(b) $\int_{(\pi/6)^{10}}^{(\pi/3)^{10}} \frac{\left(1+\tan^2\left(x^{0.1}\right)\right)^{1/2}}{x^{0.9}} dx$

Solution:

(b) Substitution: $u = x^{0.1}$, $du = 0.1x^{-0.9}dx$, $10du = x^{-0.9}dx$. New limits: $u = \pi/6$ and $u = \pi/3$.

$$\int_{(\pi/6)^{10}}^{(\pi/3)^{10}} \frac{\left(1+\tan^2\left(x^{0.1}\right)\right)^{1/2}}{x^{0.9}} dx = \int_{\pi/6}^{\pi/3} \left(1+\tan^2(u)\right)^{1/2} 10du$$

$$= \int_{\pi/6}^{\pi/3} 10 \sec u\, du$$

$$= 10 \ln|\sec u + \tan u|\Big|_{\pi/6}^{\pi/3}$$

$$= 10\ln|2+\sqrt{3}| - 10\ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| = \boxed{10\ln\left|\frac{2}{\sqrt{3}}+1\right|} \approx 7.67652$$

3. (10 points) In this question, you do not need to show work or to justify your answers.

   Parts (a) and (b) below are not related.

   (a) Suppose $f(x)$ is a continuous function defined for all real numbers x, and $A(x) = \int_0^x f(t)\,dt$.

   i. For what values of x is the graph of the curve $y = A(x)$ increasing?
   Choose one of the following:

   ☑ where $f(x) > 0$

   ☐ where $f'(x) > 0$

   ☐ where $f''(x) > 0$

   when $A'(x) > 0$

   but $A'(x) = f(x)$ by FTC I

   ii. For what values of x is the graph of the curve $y = A(x)$ concave up?
   Choose one of the following:

   ☐ where $f(x) > 0$

   ☑ where $f'(x) > 0$

   ☐ where $f''(x) > 0$

   when $A''(x) > 0$

   but $A''(x) = f'(x)$

   (b) A particle is moving along a line with velocity $v(t) = t^2 - 8t + 15$.

   During which of the following time intervals is the displacement of the particle during that interval equal to the total distance traveled during that time interval?

   Choose all that apply.

   ☑ $1 \leq t \leq 3$

   ☐ $2 \leq t \leq 4$

   ☐ $4 \leq t \leq 6$

   ☑ $5 \leq t \leq 7$

   ↑ $[a,b]$ where $\int_a^b v(t)\,dt = \int_a^b |v(t)|\,dt$

   so $[a,b]$ where $v(t) \geqslant 0$

   $v(t) = (t-3)(t-5) \geqslant 0$ for $t \leq 3$ or $t \geqslant 5$

4. (10 points) A region is bounded by the function

$$x = y\ln(y+1),$$

the x-axis and the line $x = \ln 2$. Note that when $y = 1$, $x = \ln 2$.

Find the volume of the solid of revolution obtained by rotating the region about the x-axis. Give the answer in exact form or as a decimal number with 5 significant digits.

Solution:

$$\text{Volume} = \int_0^1 2\pi y(\ln 2 - y\ln(y+1))dy$$

$$= \int_0^1 (2\pi\ln 2)ydy - \int_0^1 2\pi y^2\ln(y+1)dy \tag{1}$$

We compute the first integral.

$$\int_0^1 (2\pi\ln 2)ydy = (2\pi\ln 2)\frac{1}{2}y^2\Big|_0^1 = \pi\ln 2.$$

For the second integral in (1), we use the substitution $u = y+1$.

$$\int_0^1 2\pi y^2\ln(y+1)dy = \int_1^2 2\pi(u-1)^2\ln u\, du$$

Then we apply integration by parts.

$$\int_1^2 2\pi(u-1)^2\ln u\, du = 2\pi\frac{1}{3}(u-1)^3\ln u\Big|_1^2 - \frac{2\pi}{3}\int_1^2 (u-1)^3\frac{1}{u}du$$

$$= \frac{2}{3}\pi\ln 2 - \frac{2\pi}{3}\int_1^2 (u^3-3u^2+3u-1)\frac{1}{u}du = \frac{2}{3}\pi\ln 2 - \frac{2\pi}{3}\int_1^2 (u^2-3u+3-1/u)du$$

$$\begin{aligned}
&= \frac{2}{3}\pi\ln 2 - \frac{2\pi}{3}\left(\frac{1}{3}u^3 - \frac{3}{2}u^2 + 3u - \ln|u|\right)\Big|_1^2 \\
&= \frac{2}{3}\pi\ln 2 - \frac{2\pi}{3}\left(\frac{1}{3}\cdot 8 - \frac{3}{2}\cdot 4 + 3\cdot 2 - \ln 2\right) + \frac{2\pi}{3}\left(\frac{1}{3} - \frac{3}{2} + 3\right) \\
&= \frac{4}{3}\pi\ln 2 - \frac{5\pi}{9}
\end{aligned}$$

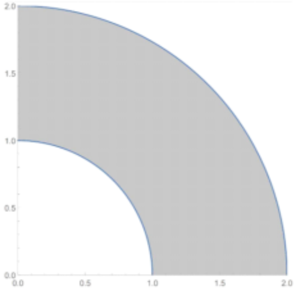
The volume is

$$\pi\ln 2 - \frac{4}{3}\pi\ln 2 + \frac{5\pi}{9} = \boxed{\pi\left(\frac{5}{9} - \frac{1}{3}\ln 2\right)} \approx 1.01947$$

5. (10 points) Suppose A is the annulus with inner radius 1 and outer radius 2, centered at $(0,0)$. Let B be the part of A in the first quadrant, as shown in the picture.

   Find the center of mass of B, assuming constant density. Give your answer in exact form or in the decimal form with at least 5 significant digits.

   Hint: It is OK to use symmetry.



Solution: The area is

$$\frac{1}{4}(\pi 2^2 - \pi 1^2) = 3\pi/4$$

$$M_x = \int_0^1 \frac{1}{2}\left(\left(\sqrt{4-x^2}\right)^2 - \left(\sqrt{1-x^2}\right)^2\right)dx + \int_1^2 \frac{1}{2}\left(\sqrt{4-x^2}\right)^2 dx$$

$$= \int_0^1 \frac{1}{2} \cdot 3dx + \int_1^2 \frac{1}{2}\left(4-x^2\right)dx$$

$$= 3/2 + \frac{1}{2}(4x - x^3/3)\Big|_1^2 = 3/2 + (4-4/3) - (2-1/6) = \frac{7}{3}$$

$$M_y = \int_0^1 x\left(\sqrt{4-x^2} - \sqrt{1-x^2}\right)dx + \int_1^2 x\sqrt{4-x^2}dx$$

$$= \int_0^2 x\sqrt{4-x^2}dx - \int_0^1 x\sqrt{1-x^2}dx$$

For the first integral, we use the substitution $u = 4 - x^2$, $du = -2xdx$, $xdx = -(1/2)du$, with the new limits $u = 4$ and $u = 0$.

$$\int_0^2 x\sqrt{4-x^2}dx = \int_4^0 u^{1/2}(-1/2)du = -\frac{1}{2} \cdot \frac{2}{3}u^{3/2}\Big|_4^0 = \frac{8}{3}$$

For the second integral, we use the substitution $u = 1 - x^2$, $du = -2xdx$, $xdx = -(1/2)du$, with the new limits $u = 1$ and $u = 0$.

$$\int_0^1 x\sqrt{1-x^2}dx = \int_1^0 u^{1/2}(-1/2)du = -\frac{1}{2} \cdot \frac{2}{3}u^{3/2}\Big|_1^0 = \frac{1}{3}$$

Hence

$$M_y = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

The center of mass is

$$\boxed{\left(\frac{28}{9\pi}, \frac{28}{9\pi}\right)} \approx (0.990297, 0.990297)$$

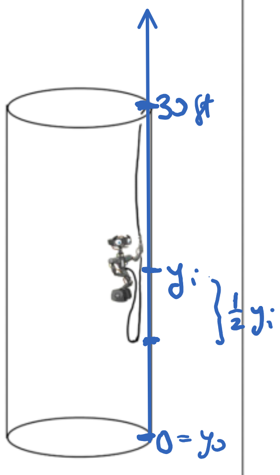
By symmetry, $M_x = M_y$ so full credit should be given if only one of the moments is computed and then the symmetry is applied.

6. (10 points) A rope weighing 0.3 pounds per foot was tied to a robot and it was used to lower the robot into a 30-foot deep well.

The robot will get out of the well by climbing up the rope at a constant speed, with the end of the rope still tied to the robot.

At the beginning of the climb, the robot weighs 20 pounds, but it will burn fuel at a constant rate and will lose 3 pounds of fuel during the climb.

Compute the work that the robot will do in climbing up to the top of the well.



Let y denote height of robot above bottom of well.

$0 \leq y \leq 30$ ft.

The robot burns fuel at a rate of $\frac{3 \text{ lbs}}{30 \text{ ft}} = 0.1$ lbs/ft

so, when the robot is y feet above ground, it will weigh $(20-0.1y)$ lbs, and it will also be lifting $\frac{1}{2}y$ ft. of rope, weighing an additional $0.3\frac{1}{2}y$ lbs.

Divide $[0,30]$ into n subintervals of length Δy

The work W_i done to move Δy ft, from y_{i-1} to y_i is

$$W_i \cong F_i \, \Delta y = \left[\left(20-0.1y_i\right)+\left(\frac{0.3}{2}y_i\right)\right]\Delta y = \left(20+0.05y_i\right)\Delta y \text{ ft-lbs.}$$

The total work is:

$$W = \lim_{n\to\infty} W_i = \int_0^{30} (20+0.05y)\,dy$$

$$= 20y + 0.025y^2 \Big|_0^{30}$$

$$= \ldots$$

$$= \boxed{622.5 \text{ ft-lb}}$$

7. (10 points) Let R be the region in the first quadrant below the parabola $y = -x^2 + 4x$.

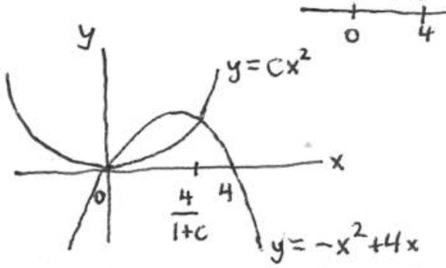
Find the value of $c > 0$ for which the graph of the parabola $y = cx^2$ divides the region R into two subregions of equal area.

Hint: Draw a picture and find the intersection points.

Graph: $y = -x^2 + 4x = -x(x-4)$

$- \; 0 \; + \; 0 \; -$

$0 \quad 4$



Intersections: $cx^2 = -x^2 + 4x$

$(1+c)x^2 - 4x = 0$

$x((1+c)x - 4) = 0$

$x = 0, \frac{4}{1+c}$

Total Area: $A = \int_0^4 -x^2 + 4x \, dx = \left(\frac{-x^3}{3} + 2x^2\right)\Big|_0^4$

$= -\frac{64}{3} + 32 = \frac{32}{3}$

Eqn for c: $\frac{1}{2} A = \frac{16}{3}$

$\frac{16}{3} = \int_0^{4/1+c} (-x^2 + 4x) - cx^2 \, dx$

$= \left(-(1+c)\frac{x^3}{3} + 2x^2\right)\Big|_0^{4/1+c}$

$= -(1+c)\frac{64}{3(1+c)^3} + 2\frac{16}{(1+c)^2}$

$\frac{16}{3} = \frac{32}{3} \frac{1}{(1+c)^2}$

$(1+c)^2 = 2$

$1 + c = \sqrt{2}$ ($c>0$, no ± needed)

$\boxed{c = \sqrt{2} - 1}$

8. (10 points) (a) Set up a definite integral for the arclength of the curve $y = 3x^3$ for $0 \leq x \leq 1$. DO NOT EVALUATE THIS INTEGRAL.

$$\frac{dy}{dx} = 9x^2$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \boxed{\int_0^1 \sqrt{1 + 81x^4}\, dx}$$

(b) Approximate the integral in part (a) using the Trapezoid Rule with $n = 3$ subintervals. Give your answer in exact form (in terms of square roots, not decimals).

$a = 0,\ b = 1,\ n = 3,\quad \Delta x = \frac{b-a}{n} = \frac{1}{3}$

$x_i = a + i\Delta x,\quad 0 \leq i \leq 3$

$x_0 = 0,\quad x_1 = \frac{1}{3},\quad x_2 = \frac{2}{3},\quad x_3 = 1$

Apply Trap Rule to $f(x) = \sqrt{1 + 81x^4}$

$$T_3 = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right)$$

$$= \frac{1}{6}\left(\sqrt{1+0} + 2\sqrt{1+1} + 2\sqrt{1+16} + \sqrt{1+81}\right)$$

$$= \boxed{\frac{1}{6}\left(1 + 2\sqrt{2} + 2\sqrt{17} + \sqrt{82}\right)}$$

$$\approx 3.52167$$

9. (10 points) Find the solution to the differential equation

$$y' = xy(y-1)$$

that satisfies the initial condition

$$y(0) = -1.$$

Give your solution in explicit form, $y = f(x)$.

Separable : $\int \frac{dy}{y(y-1)} = \int x\,dx$

partial fractions : $\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1} = \frac{A(y-1)+By}{y(y-1)}$

$y=1 \Rightarrow B=1 \;,\; y=0 \Rightarrow A=-1$

integrate : $\int \frac{-1}{y} + \frac{1}{y-1}\,dy = \int x\,dx \Rightarrow$

$-\ln|y| + \ln|y-1| = \frac{x^2}{2} + C \Rightarrow \ln\left|\frac{y-1}{y}\right| = \frac{x^2}{2} + C$

$\Rightarrow \quad 1 - \frac{1}{y} = \pm e^{C} \cdot e^{x^2/2} = D \cdot e^{x^2/2}$

$\Rightarrow \quad \frac{1}{y} = 1 - De^{x^2/2} \quad ; \quad y(0) = -1 \Rightarrow -1 = 1 - D$

$\Rightarrow \quad D = 2 \quad \Rightarrow \quad \boxed{y = \frac{1}{1-2e^{x^2/2}}}$

10. (10 points) A large vat initially contains 2000 liters of milk with 2% milk fat (by volume). Milk with 4% fat is pumped into the vat at a rate of 20 liters per minute. The milk in the vat is kept thoroughly mixed, and is pumped out of the vat, also at a rate of 20 liters per minute.

(a) What is the percentage of milk fat in the vat after 20 minutes?

Let $t =$ time (in min), $y(t) =$ volume of fat in vat (in ℓ)

$$\left.\begin{aligned}\frac{dy}{dt} &= \left(\text{incoming fat}\right) - \left(\text{outgoing fat}\right)\\ &= \left(20\,\frac{\ell\text{ milk}}{\text{min}}\right)\left(0.04\,\frac{\ell\text{ fat}}{\ell\text{ milk}}\right) - \left(20\,\frac{\ell\text{ milk}}{\text{min}}\right)\left(\frac{y\ \ell\text{ fat}}{2000\ \ell\text{ milk}}\right)\end{aligned}\right\} \Rightarrow \boxed{\frac{dy}{dt} = \frac{80-y}{100}}$$

$$\int \frac{dy}{80-y} = \int \frac{dt}{100}$$

$$-\ln|80-y| = \frac{t}{100} + C_1$$

$$|80-y| = e^{-t/100}e^{-C_1}$$

$$80-y = Ce^{-t/100} \quad (\text{where } C = \pm e^{-C_1})$$

$$y(0) = (2\%)(2000) = 40 \Rightarrow C = 40$$

$$\boxed{y = 80 - 40e^{-t/100}}$$

$$y(20) = 80 - 40e^{-1/5} \approx 47.250770\ \ell \implies$$

% fat @ $t=20$ is

$$\frac{y(20)}{2000}\cdot 100\% = \boxed{2.3625385\%}$$

(b) How many minutes after the initial time is the percentage of milk fat in the vat equal to 3%?

Want: $\frac{y(t)}{2000}\cdot 100\% = 3\%$

$$y(t) = 60\ \ell$$

$$60 = 80 - 40e^{-t/100}$$

$$e^{-t/100} = \frac{1}{2}$$

$$t = -100\ln(1/2)$$

$$= 100\ln 2$$

$$\approx \underline{69.314718}\ \text{min}$$