1. (12 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\displaystyle\int_0^{\pi/4} y\sin(y)\,dy$

Integration By Parts : $u = y$ & $dv = \sin(y)\,dy$
$du = dy$ $v = -\cos y$

$$= -y\cos y\Big|_0^{\pi/4} + \int_0^{\pi/4} \cos y\,dy$$

$$= \left[-y\cos y + \sin y\right]\Big|_0^{\pi/4}$$

$$= \left[-\frac{\pi}{4}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] - [0]$$

$$= \boxed{-\frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}}{2}} = \boxed{\frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right)}$$

(b) $\displaystyle\int_0^{\pi/6} 16\cos^2(2x)\sin^2(2x)\,dx$

$$= \int_0^{\pi/6} 16\,\frac{1+\cos(4x)}{2}\,\frac{1-\cos(4x)}{2}\,dx$$

$$= \int_0^{\pi/6} 4\left(1 - \cos^2(4x)\right)dx$$

$$= \int_0^{\pi/6} 4\sin^2(4x)\,dx$$

OR $\displaystyle\int_0^{\pi/6} 4\,\underbrace{\left(2\sin(2x)\cos(2x)\right)}^2\,dx = \int_0^{\pi/6} 4\sin^2(4x)\,dx.$

$$= \int_0^{\pi/6} 4\,\frac{1-\cos(8x)}{2}\,dx$$

$$= \left[2x - \frac{1}{4}\sin(8x)\right]\Big|_0^{\pi/6}$$

$$= \left[\frac{\pi}{3} - \frac{1}{4}\sin\left(\frac{4\pi}{3}\right)\right] - [0]$$

$$= \frac{\pi}{3} - \frac{1}{4}\left(\frac{-\sqrt{3}}{2}\right)$$

$$= \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{8}}$$

2. (14 points) Answer the following two unrelated questions. Show your work and box your answer.

(a) Evaluate the integral: $\int \ln(x^2+1)\,dx$

Integration by Parts:

$u = \ln(x^2+1) \qquad dv = dx$

$du = \frac{2x}{x^2+1}\,dx \qquad v = x$

$= x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}\,dx$

$= x\ln(x^2+1) - \int \frac{2(x^2+1)-2}{x^2+1}\,dx$

$= x\ln(x^2+1) - \int 2 - \frac{2}{x^2+1}\,dx$

$= \boxed{x\ln(x^2+1) - 2x + 2\arctan(x) + C}$

OR: $\int \frac{2x^2}{x^2+1}\,dx \qquad x = \tan\theta, \quad dx = \sec^2\theta\,d\theta$

$= \int \frac{2\tan^2\theta}{\cancel{\sec^2\theta}}\,\cancel{\sec^2\theta}\,d\theta$

$= 2\int \sec^2\theta - 1\,d\theta$

$= 2\tan\theta - 2\theta + C$

$= 2x - 2\arctan x + C$

(b) The acceleration and the initial velocity of a object moving on a straight line are given by:

$$a(t) = 2t + 6 \;\; m/s^2 \quad \text{and} \quad v(0) = -7 \;\; m/s$$

Find the <u>**total**</u> **distance** traveled by the particle from $t = 0$ to $t = 2$ seconds.

$a(t) = 2t + 6 \implies v(t) = t^2 + 6t + C$

Since $v(0) = -7$, $C = -7$

So: $v(t) = t^2 + 6t - 7 \;\; m/s = (t+7)(t-1)$

Total dist. $= \int_0^2 |t^2 + 6t - 7|\,dt$

$= \int_0^1 (-t^2 - 6t + 7)\,dt + \int_1^2 (t^2 + 6t - 7)\,dt$

$= \underbrace{\left(-\frac{1}{3}t^3 - 3t^2 + 7t\right)\Big|_0^1}_{} + \underbrace{\left(\frac{1}{3}t^3 + 3t^2 - 7t\right)\Big|_1^2}_{}$

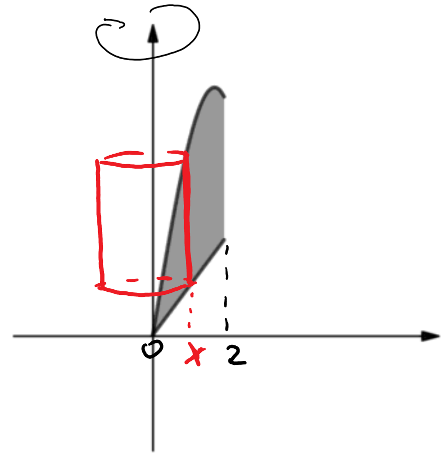
$= \frac{11}{3} + \left(\frac{2}{3} - \frac{-11}{3}\right)$

$= \boxed{8}$ meters

3. Consider the region enclosed by the graphs $y = 9x - x^3$, $y = 2x$, $x = 0$, and $x = 2$ pictured below.

(a) (4 points) **Set up** an integral that represents the volume of the solid formed by rotating this region about the y-axis. (Do not compute the volume).

Note: We cannot solve $y = 9x - x^3$ for x in terms of y so we must set up our integrals in x. This means vertical rectangles, so shells for part (a) & washers in part (b)



(a) SHELLS:

$$\boxed{V_1 = \int_0^2 2\pi x \left[(9x - x^3) - (2x)\right] dx}$$

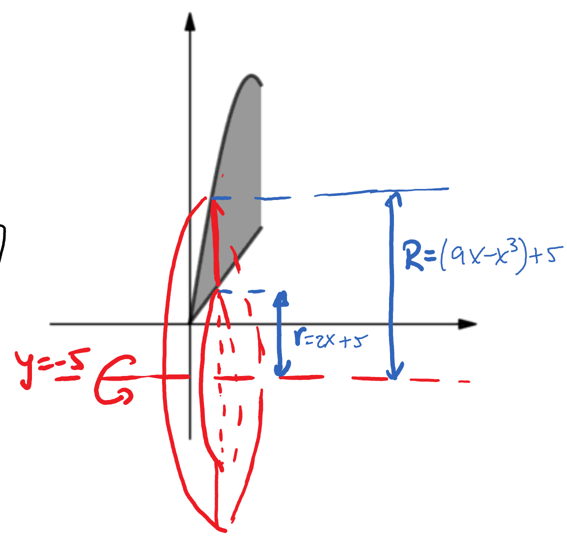
$$= \int_0^2 2\pi x \,(7x - x^3)\, dx$$

(b) (4 points) **Set up** an integral that represents the volume of the solid formed by rotating this region about the line $y = -5$. (Do not compute the volume).

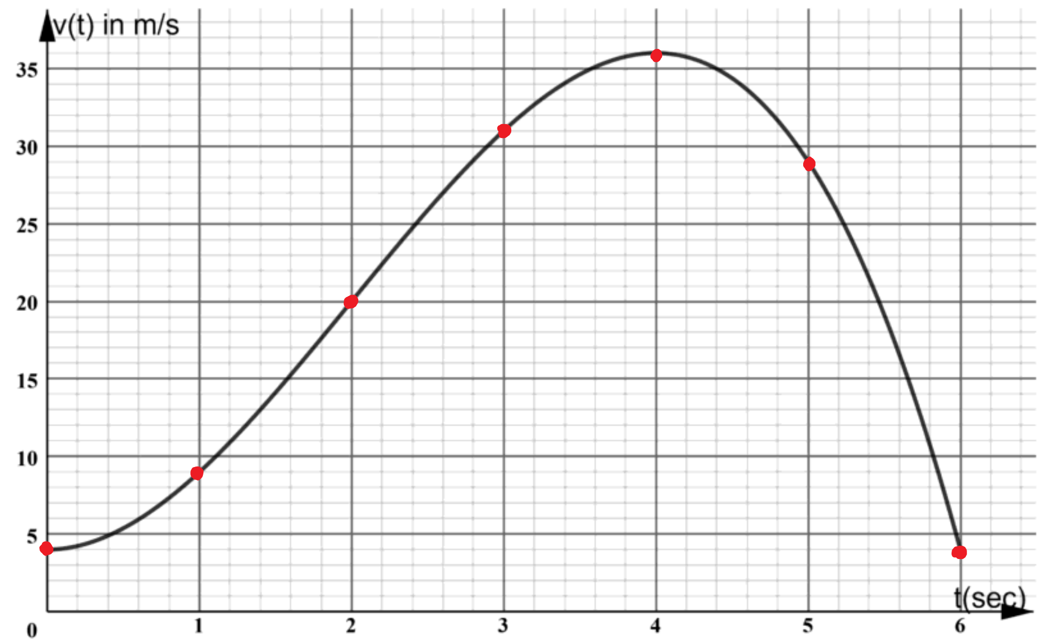
Washers:

$$V_2 = \int_0^2 \pi R^2 - \pi r^2 \; dx$$

$$= \boxed{\int_0^2 \pi \left[9x - x^3 + 5\right]^2 - \pi \left[2x + 5\right]^2 dx}$$



4. (7 points) The graph below shows the instantaneous velocity $v(t)$ (in meters per second) of an object moving along a straight line, as a function of time t (in seconds).



Use Simpsons Rule with $n = 6$ subintervals to approximate the **average velocity** v_{ave} of the object from $t = 0$ to $t = 6$ seconds.

$$V_{ave} = \frac{1}{6}\int_0^6 v(t)\,dt$$

$$\cong S_6 = \frac{1}{6}\,\frac{1}{3}\left[V(0)+4V(1)+2V(2)+4V(3)+2V(4)+4V(5)+V(6)\right]$$

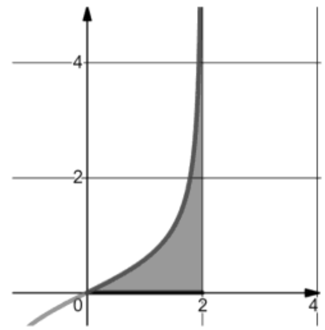
$$= \frac{1}{6}\cdot\frac{1}{3}\cdot\left[4+4(9)+2(20)+4(31)+2(36)+4(29)+4\right]$$

$$= \frac{1}{18}\left[4+36+40+124+72+116+4\right] = \frac{1}{18}\left[396\right]$$

$$= \boxed{22}\ m/s$$

5. Let $\mathcal{R}$ be the region in the first quadrant which is shown below, and it is described by:

$$0 \le y \le \frac{x}{\sqrt{4-x^2}}, \qquad 0 \le x < 2$$



Note that $f(x) = \frac{x}{\sqrt{4-x^2}}$ has a vertical asymptote; use limits for improper integrals as needed, and determine if they converge or diverge.

(a) (6 points) Compute the **area** of this region $\mathcal{R}$.

$$A = \int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx \qquad u = 4-x^2,\quad du = -2x\,dx$$

$$= \int_4^0 \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\right)du$$

$$= \frac{1}{2}\int_0^4 \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\lim_{a\to 0^+}\int_a^4 u^{-1/2}\,du.$$

$$= \frac{1}{\cancel{2}}\lim_{a\to 0^+} \cancel{2}\sqrt{u}\Big|_a^4 = \lim_{a\to 0^+}\left(\sqrt{4}-\sqrt{a}\right)$$

$$= \boxed{2}$$

[Alternative method (harder): Trig Sub]

(b) (7 points) Compute the x-coordinate, $\bar{x}$, of its centroid (center of mass).

$$\bar{x} = \frac{1}{2}\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx \qquad x = 2\sin\theta,\quad dx = 2\cos\theta\,d\theta$$

$x=0 \Rightarrow \theta = 0$
$x=2 \Rightarrow \theta = \arcsin(1) = \frac{\pi}{2}$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{4\sin^2\theta}{\cancel{2\cos\theta}}\,\cancel{2\cos\theta}\,d\theta$$

$$= \int_0^{\pi/2} \cancel{2}\,\frac{1-\cos 2\theta}{\cancel{2}}\,d\theta$$

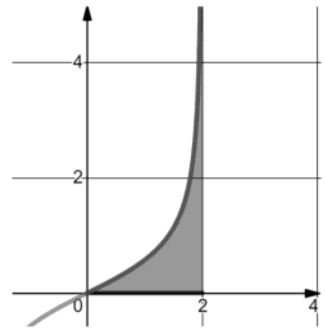
$$= \left[\theta - \frac{1}{2}\sin(2\theta)\right]\Big|_0^{\pi/2}$$

$$= \left[\frac{\pi}{2} - \frac{1}{2}\underbrace{\cancel{\sin\pi}}_{0}\right] - [0]$$

$$= \boxed{\frac{\pi}{2}}$$

(problem continues on the next page)

(c) (8 points) *Recall the region $\mathcal{R}$ from the previous page, bounded above by* $y = \dfrac{x}{\sqrt{4-x^2}}$, for $0 \le x < 2$.



Use limits for improper integrals as needed, and determine if they converge or diverge.

Compute the y-coordinate, $\bar{y}$, of the centroid of $\mathcal{R}$.

$$\bar{y} = \frac{1}{2}\int_0^2 \frac{1}{2}\left(\frac{x}{\sqrt{4-x^2}}\right)^2 dx$$

$$= \frac{1}{4}\int_0^2 \frac{x^2}{4-x^2}\,dx.$$

$$= \frac{1}{4}\lim_{b\to 2^-}\int_0^b \frac{x^2}{4-x^2}\,dx$$

$$= \frac{1}{4}\lim_{b\to 2^-}\left[-x+\ln\left|\frac{2+x}{2-x}\right|\right]\Bigg|_0^b$$

$$= \frac{1}{4}\left[\lim_{b\to 2^-}\left[-b+\ln\left|\frac{2+b}{2-b}\right|\right] - [0]\right]$$

$$= \frac{1}{4}\left[-2+\underbrace{\lim_{b\to 2^-}\ln\left|\frac{2+b}{2-b}\right|}_{\infty}\right]$$

$$\boxed{\text{DIVERGES}}$$

$$\int \frac{x^2}{4-x^2}\,dx$$

$$= \int -1 + \frac{4}{4-x^2}\,dx$$

$$= -x + \int \frac{4}{(2-x)(2+x)}\,dx$$

P.F.: $\dfrac{4}{(2-x)(2+x)} = \dfrac{A}{2-x} + \dfrac{B}{2+x}$

$4 = A(2+x) + B(2-x)$

$x=2:\quad 4 = A(4) \Rightarrow A = 1$

$x=-2:\quad 4 = B(4) \Rightarrow B = 1.$

$$= -x + \int \frac{1}{2-x} + \frac{1}{2+x}\,dx$$

$$= -x - \ln|2-x| + \ln|2+x| + C$$

$$= -x + \ln\left|\frac{2+x}{2-x}\right| + C$$

Remark: $\int \frac{x^2}{4-x^2}\,dx$ could also be computed using a trig sub with $x = 2\sin\theta$ $dx = 2\cos\theta$ (answ: $-x + 2\ln\left|\frac{x+2}{\sqrt{4-x^2}}\right|$) (or $x = 2\cos\theta$)

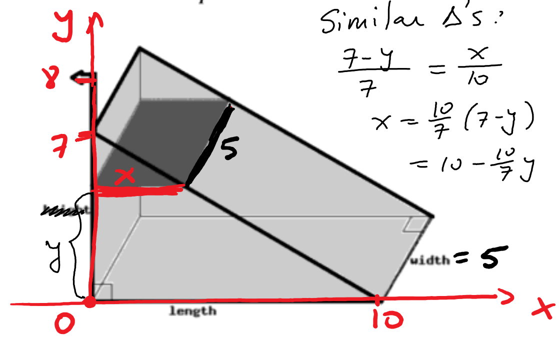
However, $x = 2\sec\theta$ is not correct because the bounds are $0 \le x \le 2$ so $\sec\theta = \frac{x}{2}$ would be < 1 which is not possible. You'd also get an impossible right triangle.

6. (8 points) A tank of the shape shown in the picture, with height=7m, length=10m, and width=5m, is full of water. Water weighs 1000 kg/m^3, and the gravitational acceleration is $g = 9.8$ m/s^2.

   Set up (do not evaluate) an integral equal to the work required to pump all the water out of the tank through a spout that is 1 m above the top of the tank.

   Specify the meaning of your variable of integration, either in words or on the picture.

With y = height above bottom:



Similar Δ's:

$$\frac{7-y}{7} = \frac{x}{10}$$

$$x = \frac{10}{7}(7-y)$$

$$= 10 - \frac{10}{7}y$$

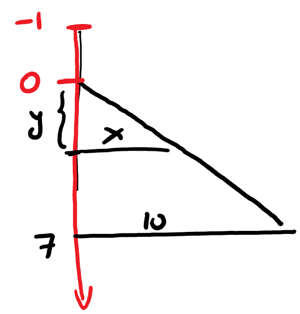
$$W = \int_0^7 9.8\,(1000)\,(8-y)\,(5x\,dy)$$

$$= \boxed{\int_0^7 9800\,(8-y)\,\frac{50}{7}(7-y)\,dy}$$

OR

w/ y = depth below top of tank:

$$\boxed{W = \int_0^7 9800\,(y+1)\left(\frac{50}{7}y\right)dy}$$



$$\frac{y}{7} = \frac{x}{10}$$

$$x = \frac{10}{7}y.$$

7. (a) (4 points) Write down an integral equal to the **arclength** $L(t)$ of the portion of the curve:

$$y = e^{x^2}, \text{ from } x = 0 \text{ to } x = t.$$

$$dy/dx = e^{x^2} \cdot 2x$$

$$L(t) = \int_0^t \sqrt{1 + (e^{x^2} \cdot (2x))^2}\,dx = \int_0^t \sqrt{1 + 4x^2 e^{2x^2}}\,dx$$

(b) (4 points) At what rate is $L(t)$ increasing when $t = 1$?

$$L'(t) = \sqrt{1 + 4t^2 e^{2t^2}}$$

$$\boxed{L'(1) = \sqrt{1 + 4e^2}}$$

8. (10 points) Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{xy+y}{2\ln(y)}$$

that satisfies the initial condition $y(1) = e^2$. Give your solution in explicit form, $y = f(x)$.

$$\frac{dy}{dx} = \frac{x+1}{2} \frac{y}{\ln y}$$

$$\int \frac{\ln y}{y}\, dy = \int \frac{x+1}{2}\, dx$$

$u = \ln y$
$du = \frac{1}{y} dy$

$$\int u\, du = \frac{1}{2}\left(\frac{x^2}{2}+x\right)+C$$

$$\cancel{\tfrac{1}{2}} u^2 = \cancel{\tfrac{1}{2}}\left(\frac{x^2}{2}+x\right)+C$$

$$(\ln y)^2 = \frac{x^2}{2}+x+C_1$$

$y(1)=e^2 \Rightarrow \underbrace{(\ln e^2)^2}_{4} = \frac{1}{2}+1+C_1 \Rightarrow C_1 = 4-\frac{1}{2}-1 = 3-\frac{1}{2}$

$C_1 = 5/2$

$$(\ln y)^2 = \frac{x^2}{2}+x+\frac{5}{2}$$

$$\ln y = \pm\sqrt{\frac{x^2}{2}+x+\frac{5}{2}}$$

$y(1)=e^2 \Rightarrow$ we need the $+$

$$\ln y = \sqrt{\frac{x^2}{2}+x+\frac{5}{2}}$$

$$\boxed{y = e^{\sqrt{x^2/2+x+5/2}}}$$

9. A 2000 L tank is full of a mixture of water and salt, with 500 grams of salt initial dissolved in the tank. Fresh water (with NO salt) is pumped into the tank at a rate of 20 L/s. The mixture is kept stirred and is pumped out at a rate of 40 L/s. (This means the tank is losing volume at a rate of 20 - 40 = -20 L/s).

   (a) (1 point) Give the linear function $V(t) = at + b$ for the volume in liters after t seconds.

   $$\boxed{V(t) = -20t + 2000}$$

   (b) (4 points) Let $y(t)$ be the amount of salt in grams in the tank after t seconds. Write down the differential equation AND initial condition satisfied by $y(t)$. Do not solve anything yet.

   $$\boxed{\frac{dy}{dt} = 0 - \left(\frac{y}{-20t+2000}\right)(40)} \leftarrow \text{simplifies to } \frac{dy}{dt} = \frac{4\cancel{0}y}{2\cancel{0}t - 200\cancel{0}} = \frac{2y}{t-100}.$$

   $$\boxed{y(0) = 500}$$

   (c) (6 points) Solve the differential equation to find $y(t)$. Show work. Simplify and box your answer.

   $$\frac{dy}{dt} = \frac{2y}{t-100} \Rightarrow \int \frac{1}{y}\,dy = \int \frac{2}{t-100}\,dt$$

   $$\ln|y| = 2\ln|t-100| + c$$

   $(y > 0)$ $$y = e^{2\ln|t-100| + c} = C_1 e^{2\ln|t-100|} = C_1 \cancel{e}^{\ln(|t-100|^2)}$$

   $$y = C_1 |t-100|^2$$

   $y(0) = 500$: $$500 = C_1 |-100|^2 = C_1 (10000) \Rightarrow C_1 = 5\cancel{0}\cancel{0}/1000\cancel{0}\cancel{0} = \frac{1}{20}.$$

   $$\boxed{y = \frac{1}{20}|t-100|^2}$$

   (d) (1 point) How many grams of salt are left in the tank after 60 seconds? Simplify your answer.

   $$y(60) = \frac{1}{20}|60-100|^2 = \frac{1}{20}|-40|^2 = \frac{160\cancel{0}}{2\cancel{0}} = \boxed{80 \text{ grams}}$$