SOLUTIONS

1. (14 points)

(a) (7 points) Evaluate the integral $\int \frac{1}{x^3-4x^2}\,dx$. Show your work, and box your answer.

$$\frac{1}{x^3-4x^2} = \frac{1}{x^2(x-4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4} \quad \text{(Partial Fractions)}$$

$$1 = Ax(x-4) + B(x-4) + Cx^2$$

Solving for A, B, C : $B = -\frac{1}{4}$, $C = 1/16$, $A = -\frac{1}{16}$

$$\therefore \int \frac{1}{x^3-4x^2}\,dx = \int \frac{-1/16}{x} + \frac{-1/4}{x^2} + \frac{1/16}{x-4}\,dx$$

$$= -\frac{1}{16}\ln|x| - \frac{1}{4}\left(\frac{-1}{x}\right) + \frac{1}{16}\ln|x-4| + C$$

$$= \boxed{-\frac{1}{16}\ln|x| + \frac{1}{4x} + \frac{1}{16}\ln|x-4| + C}$$

$$= \frac{1}{4x} + \frac{1}{16}\ln\left|\frac{x-4}{x}\right| + c = \frac{1}{4x} + \ln\left(\sqrt[16]{\left|\frac{x-4}{x}\right|}\right) + C$$

(b) (7 points) Evaluate the following improper integral, if it converges, or show why it diverges.

$$\int_0^\infty \frac{e^x}{1+e^{2x}}\,dx$$

$$\int_0^\infty \frac{e^x}{1+e^{2x}}\,dx \underset{\substack{u=e^x \\ du=e^x dx}}{=} \int_1^\infty \frac{1}{1+u^2}\,du = \lim_{t\to\infty}\int_1^t \frac{1}{1+u^2}\,du$$

$$= \lim_{t\to\infty} (\arctan u)\Big|_1^t$$

$$= \lim_{t\to\infty}\left[\arctan t - \underbrace{\arctan 1}\right]$$

$$= \underbrace{\lim_{t\to\infty} (\arctan t)} - \frac{\pi}{4}$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$\boxed{\text{Integral converges to } \frac{\pi}{4}}$.

2. (14 points)

(a) (7 points) Evaluate $\int_0^{\sqrt{3}} x\tan^{-1}(x)dx$.
Give your answer in exact form (in terms of square roots and/or multiples of π).

Applying Integration By Parts with $u = \tan^{-1}x$, $du = \frac{1}{1+x^2}dx$, $dv = x\,dx$, $v = \frac{x^2}{2}$

we get:

$$= \frac{x^2}{2}\tan^{-1}x\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{1+x^2}\,dx$$

$$\frac{x^2}{1+x^2} = \frac{(1+x^2)-1}{1+x^2} = 1 - \frac{1}{1+x^2}$$
(or: long division)

$$= \left(\frac{3}{2}\tan^{-1}\sqrt{3} - 0\right) - \frac{1}{2}\int_0^{\sqrt{3}} 1 - \frac{1}{1+x^2}\,dx$$

$$= \frac{3}{2}\,\frac{\pi}{3} - \frac{1}{2}\left(x - \tan^{-1}x\right)\Big|_0^{\sqrt{3}}$$

$$= \frac{3\pi}{6} - \frac{1}{2}\sqrt{3} + \frac{1}{2}\,\frac{\pi}{3} = \frac{4\pi}{6} - \frac{\sqrt{3}}{2} = \boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}}$$

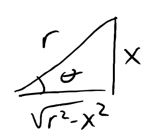
(b) (7 points) Find the function $f(x)$ if $f'(x) = \dfrac{1}{(r^2-x^2)^{3/2}}$ and $f(0) = 0$.
The constant r should appear in your answer.

Applying Trig Sub with $x = r\sin\theta$, $dx = r\cos\theta\,d\theta$, we get:

$$\int \frac{1}{(r^2-x^2)^{3/2}}\,dx = \int \frac{1}{(r^2 - r^2\sin^2\theta)^{3/2}}\, r\cos\theta\,d\theta = \int \frac{1}{r^3\cos^3\theta}\, r\cos\theta\,d\theta$$

$$= \frac{1}{r^2}\int \sec^2\theta\,d\theta$$

$$= \frac{1}{r^2}\tan\theta + C$$



$$= \frac{1}{r^2}\,\frac{x}{\sqrt{x^2-r^2}} + C$$

$f(x) = \frac{1}{r^2}\frac{x}{\sqrt{x^2-r^2}} + C$ and $f(0)=0 \Rightarrow C = 0$.

$$\therefore \boxed{f(x) = \frac{x}{r^2\sqrt{x^2-r^2}}}$$

3. (13 points) The velocity of a particle is given by $v(t) = \sin^3(\pi t)$ ft/sec where t is in seconds.

(a) (7 points) Assume the initial position of the particle is $s(0) = 0$ ft.
Find the function $s(t)$ for the position of the particle at time t.

$$S(t) = \int \sin^3(\pi t)\,dt \qquad \text{with } S(0) = 0$$

$$= \int (1-\cos^2(\pi t))\sin(\pi t)\,dt \qquad u = \cos(\pi t)$$
$$du = -\pi \sin(\pi t)\,dt$$

$$= -\frac{1}{\pi}\int 1 - u^2\,du$$

$$= \frac{1}{\pi}\left(\frac{u^3}{3} - u\right) + C = \frac{1}{\pi}\left(\frac{\cos^3(\pi t)}{3} - \cos(\pi t)\right) + C$$

$$S(0) = 0 \iff 0 = \frac{1}{\pi}\left(\frac{1}{3} - 1\right) + C \iff C = \frac{2}{3\pi}$$

$$\therefore S(t) = \frac{1}{\pi}\left(\frac{\cos^3(\pi t)}{3} - \cos(\pi t)\right) + \frac{2}{3\pi}$$
$$= \boxed{\frac{1}{3\pi}\cos^3(\pi t) - \frac{1}{\pi}\cos(\pi t) + \frac{2}{3\pi}}$$

(b) (6 points) Find the total distance traveled by the particle from $t = 0$ to $t = \frac{3}{2}$ seconds.

We want: $\int_0^{3/2} |v(t)|\,dt = \int_0^{3/2} |\sin^3(\pi t)|\,dt$

Over the interval $[0, 3/2]$: $\sin^3(\pi t) = 0$ at $t = 1$
it's ≥ 0 on $[0,1]$ and ≤ 0 on $[1, 3/2]$

so we get: $\int_0^{3/2} |\sin^3(\pi t)|\,dt = \int_0^1 \sin^3(\pi t)\,dt + \int_1^{3/2} -\sin(\pi t)\,dt$

Using the antiderivative found above:

$$= \frac{1}{\pi}\left[\frac{\cos^3(\pi t)}{3} - \cos(\pi t)\right]\Big|_0^1 - \frac{1}{\pi}\left[\frac{\cos^3(\pi t)}{3} - \cos(\pi t)\right]\Big|_1^{3/2}$$

$\cos(\pi) = -1$
$\cos\left(\frac{3\pi}{2}\right) = 0$

$$= \frac{1}{\pi}\left[\left(\frac{-1}{3} + 1\right) - \left(\frac{1}{3} - 1\right)\right] - \frac{1}{\pi}\left[(0 - 0) - \left(\frac{1}{3} - 1\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{4}{3}\right) - \frac{1}{\pi}\left(-\frac{2}{3}\right) = \frac{6}{3\pi} = \boxed{\frac{2}{\pi}\text{ feet}}$$

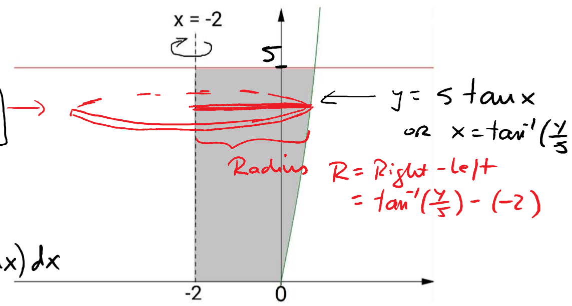
4. (14 points) Let R be the region enclosed by: the x-axis, the line $y = 5$, the line $x = -2$, and the portion of the curve $y = 5\tan(x)$ between $x = 0$ and $x = \pi/4$. The region R is rotated around the line $x = -2$ to form a solid of revolution. The units are meters. In parts (b) and (c) take g to be 9.8 m/sec^2 and take the density of water to be 1000 kg/m^2.

   Write each of the following in terms of integrals, but do **not** evaluate the integrals.

   (a) (7 points) the volume of the resulting container;

Disks (in y) : $V = \int_0^5 \pi\,(\text{radius})^2\,dy$

$$= \boxed{\pi \int_0^5 \left(\arctan\left(\frac{y}{5}\right)+2\right)^2 dy}$$



OR Shells (in x) : $V = \int_{-2}^{0} 2\pi R_1 h_1\,dx + \int_0^{\pi/4} 2\pi R_2 h_2\,dx$

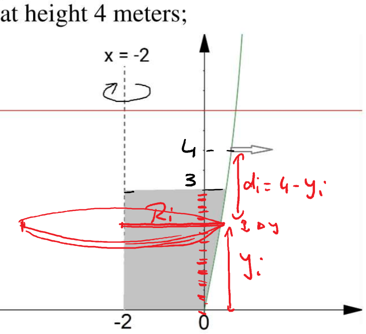
$$= \underbrace{\int_{-2}^{0} 2\pi(x+2)5\,dx}_{\text{volume of cylinder: } 20\pi} + \int_0^{\pi/4} 2\pi(x+2)(5-5\tan x)\,dx$$

   (b) (4 points) the amount of work (in Joules) required to empty the container of water, if water is filled up to the level of 3 meters, and there's an outtake pipe at height 4 meters;

Slice the water (interval [0, 3]) into n horizontal slices. The i^{th} slice is a disk of thickness Δy and we have:

Weight of slice : $F_i = (9.8)(1000)\pi\left(2+\tan^{-1}\left(\frac{y_i}{5}\right)\right)^2 \Delta y$

Distance to lift slice : $d_i = 4 - y_i$



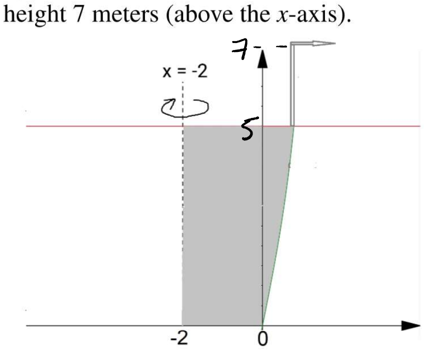
Work $W = \lim_{n\to\infty} \sum_{i=1}^{n} 9800\pi\left(2+\tan^{-1}\left(\frac{y_i}{5}\right)\right)^2 (4-y_i)\,\Delta y$

$$\therefore \boxed{W = \int_0^3 9800\pi\left[2+\tan^{-1}\left(\frac{y}{5}\right)\right]^2 (4-y)\,dy}$$

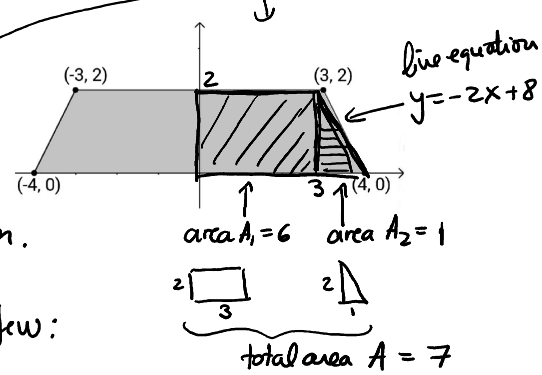
   (c) (3 points) the amount of work (in Joules) required to empty the container of water if the container is filled to the top with water and the outtake pipe is at height 7 meters (above the x-axis).

A very similar process yields:

$$\int_0^5 9800\pi\left[2+\tan^{-1}\left(\frac{y}{5}\right)\right]^2 (7-y)\,dy$$



5. (10 points) Find the coordinates $(\bar{x}, \bar{y})$ for the center of mass of the region shown below.



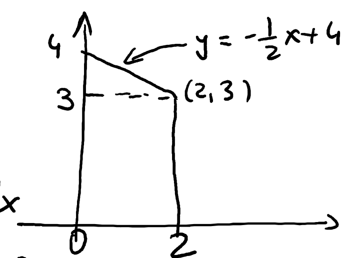
By symmetry: $\boxed{\bar{x} = 0}$

Note: We will compute $\bar{y}$ for the right half of the region since, by symmetry again, it will be the same as $\bar{y}$ for the entire region.

There are many ways to compute $\bar{y}$. Here are a few:

Method I: $$\bar{y} = \frac{1}{\text{area}} M_x = \frac{1}{7}\left[\underbrace{\int_0^3 \frac{1}{2}(2)^2 dx}_{6} + \int_3^4 \frac{1}{2}(-2x+8)^2 dx\right]$$
$$= \frac{1}{7}\left[6 + \left(\frac{2}{3}x^3 - 8x^2 + 32x\right)\Big|_3^4\right]$$
$$= \frac{1}{7}\left[6 + \frac{2}{3}\right] = \boxed{\frac{20}{21}}$$

Method II: Switch the axes and compute $\bar{x}$ for the resulting region:



$$\text{original } \bar{y} = \text{new } \bar{x} = \frac{1}{\text{area}} \int_0^2 x f(x) dx = \frac{1}{7}\int_0^2 x\left(-\frac{1}{2}x+4\right)dx$$
$$= [\ldots] = \frac{1}{7}\left(-\frac{1}{6}x^3 + 2x^2\right)\Big|_0^2$$
$$= [\ldots] = \frac{1}{7}\left(\frac{20}{3} - 0\right) = \boxed{\frac{20}{21}}$$

Method III: Decompose the region into $\boxed{A_1}$ & A_2 and use

$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2} = \frac{(6)(1) + (1)\left[\frac{1}{(1)}\int_3^4 \frac{1}{2}(-2x+8)^2 dx\right]}{6+1}$$
$$= \frac{(6)(1) + (1)\left(\frac{2}{3}\right)}{7} = \frac{20/3}{7} = \boxed{\frac{20}{21}}$$

Either way, the answer is:

$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{20}{21}\right)}$$

6. (10 points) Find the explicit solution $y = y(x)$ to the initial value problem

$$\frac{dy}{dx} = y^2 e^{\sqrt{x}}\ ,\ y(0) = \frac{1}{5}.$$

Separate the variables and integrate:

$$\int \frac{1}{y^2}\,dy = \int e^{\sqrt{x}}\,dx$$

Rationalizing substitution

$u = \sqrt{x} \Rightarrow x = u^2$

$dx = 2u\,du$

$$-\frac{1}{y} = 2\int u e^u\,du$$

Integration by Parts

$w = u \quad dv = e^u du$

$dw = du \quad v = e^u$

$$-\frac{1}{y} = 2ue^u - 2\int e^u\,du$$

$$-\frac{1}{y} = 2ue^u - 2e^u + C$$

$$-\frac{1}{y} = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

$y(0) = \frac{1}{5} \Rightarrow \qquad -5 = 0 - 2 + C \Rightarrow C = -3$

$$\therefore -\frac{1}{y} = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} - 3$$

$$\boxed{y = \frac{-1}{2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} - 3} = \frac{1}{3 + 2e^{\sqrt{x}} - 2\sqrt{x}e^{\sqrt{x}}}}$$

7. (13 points) Suppose you **drop** a stone of mass m from a great height in the earth's atmosphere, and the only forces acting on the stone are the earth's gravitational attraction and a retarding force due to air resistance, which is proportional to the velocity v. Take downward to be the positive direction. Then, since $F = ma$ and $a = dv/dt$, we have the differential equation:

$$m\frac{dv}{dt} = mg - kv,$$

where k is a positive constant. Suppose that the mass is $m = 1$ kg, and take $g = 9.8$ m/sec^2.

(a) (6 points) Solve the differential equation to find a formula for $v(t)$. Your answer will involve k.

With $m = 1$kg and $g = 9.8$, we have: $\frac{dv}{dt} = 9.8 - kv$

Separating the variables and integrating: $\int \frac{1}{9.8 - kv}\, dv = \int 1\, dt$

$$-\frac{1}{k}\ln|9.8 - kv| = t + C$$

Initial condition is: $v(0) = 0$ ("drop"), so $-\frac{1}{k}\ln 9.8 = C$

Replacing in the equation: $-\frac{1}{k}\ln|9.8 - kv| = t - \frac{1}{k}\ln 9.8$

Solving for v: $\ln|9.8 - kv| = -kt + \ln 9.8 \Rightarrow |9.8 - kv| = e^{-kt} \cdot e^{\ln 9.8} = 9.8e^{-kt}$

$$9.8 - kv = \pm 9.8e^{-kt}$$

$$kv = 9.8 \pm 9.8e^{-kt} \Rightarrow v = \frac{9.8}{k}(1 \pm e^{-kt})$$

Using the initial condition $v(0) = 0$ to fix the sign, we get

$$\boxed{v = \frac{9.8}{k}(1 - e^{-kt})} \text{ (in m/s.)}$$

(b) (3 points) Compute the terminal velocity v_∞ (the limiting velocity as $t \to \infty$). Your answer will involve the positive constant k.

$$v_\infty = \lim_{t \to \infty} \frac{9.8}{k}(1 - e^{-kt}) = \frac{9.8}{k}(1 - 0)$$

$$= \boxed{\frac{9.8}{k} \text{ m/s}}.$$

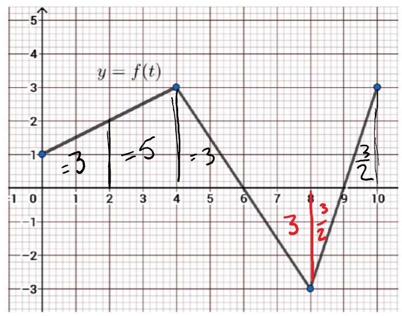
(c) (4 points) If $v_\infty = 70$ m/sec, find the speed of the stone after 3 sec.

$$70 = \frac{9.8}{k} \Rightarrow k = \frac{9.8}{70} = 0.14$$

From (a) $v = \frac{9.8}{k}(1 - e^{-kt}) = 70\left(1 - e^{-\frac{9.8}{70}t}\right)$

At $t = 3$ sec: $v = 70\left(1 - e^{-\frac{9.8}{70}(3)}\right) = \boxed{70\left(1 - e^{-0.42}\right)}$ ($\cong 24$ m/s)

8. (12 points) Suppose that the graph of f is as shown:



(a) (4 points) Compute the average value of this function over the interval $[0, 10]$.

$$f_{ave} = \frac{1}{10}\int_0^{10} f(x)\,dx =$$

$$= \frac{1}{10}\left(\text{signed area between } y = f(x) \text{ and the } x\text{-axis from } x=0 \text{ to } x=10\right)$$

$$= \frac{1}{10}\left(3 + 5 + \cancel{3} - \cancel{3} - \cancel{\tfrac{3}{2}} + \cancel{\tfrac{3}{2}}\right)$$

$$= \frac{1}{10}(8)$$

$$= \boxed{\frac{8}{10} = 0.8}$$

(b) Define a new function $A(x) = \int_x^{x^3} f(t)\,dt$, where f is the same function as above.

i. (2 points) Compute $A(2)$.

$$A(2) = \int_2^8 f(t)\,dt = 5 + 3 - 3 = \boxed{5}$$

ii. (6 points) Compute $A'(2)$.

$$A'(x) = \frac{d}{dx}\int_x^{x^3} f(t)\,dt = \frac{d}{dx}\left(\int_x^0 f(t)\,dt + \int_0^{x^3} f(t)\,dt\right)$$

$$= \underbrace{\frac{d}{dx}\left(-\int_0^x f(t)\,dt\right)}_{FTC\ I} + \underbrace{\frac{d}{dx}\left(\int_0^{x^3} f(t)\,dt\right)}_{FTC\ I\ +\ Chain}$$

$$= -f(x) + \quad + f(x^3)\cdot(3x^2)$$

$$\therefore A'(x) = -f(x) + 3x^2 f(x^3)$$

$$A'(2) = -f(2) + 12f(8) \qquad \text{(Reading } y\text{-values on graph)}$$

$$= -2 + 12(-3) = \boxed{-38}$$