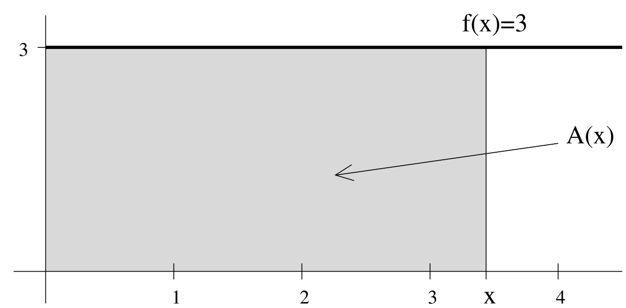
Name ________________________ Quiz Section ______

Introduction

Many interesting and useful functions can be defined as the area under some other function. There is a very nice relationship between the original function and the area function. We will explore that relationship in this worksheet.

Area Functions

1a Define $A(x)$ to be the **area** bounded by the x-axis and the function $f(x) = 3$ between the y-axis and the vertical line at x. (See the diagram.)



$A(1) =$ __________ $A(2) =$ __________

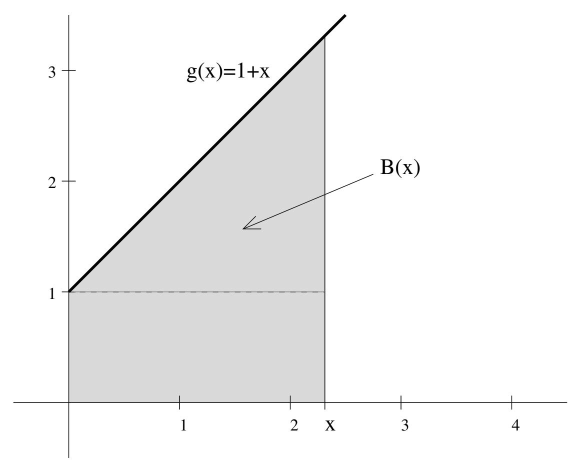
$A(3) =$ __________ $A(4) =$ __________

and, in general,

$A(x) =$ ____________________ (a formula)

Shade the region whose area is $A(3) - A(1)$.

1b Define $B(x)$ to be the **area** bounded by the x-axis and the function $g(x) = 1 + x$ between the y-axis and the vertical line at x. (See the diagram.)



$B(1) =$ __________ $B(2) =$ __________

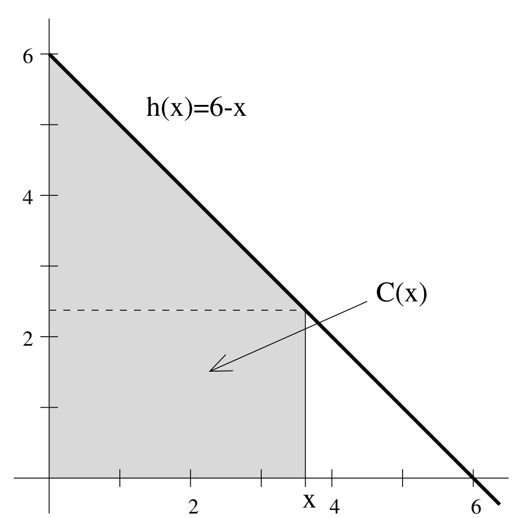
$B(3) =$ __________ $B(4) =$ __________

and, in general,

$B(x) =$ ____________________ (a formula)
(Hint: think triangle + rectangle)

Shade the region whose area is $B(3) - B(1)$.

1c Define $C(x)$ to be the **area** bounded by the x-axis and the function $h(x) = 6 - x$ between the y-axis and the vertical line at x. (See the diagram.)



$C(1) =$ __________ $C(2) =$ __________

$C(3) =$ __________ $C(4) =$ __________

and, in general,

$C(x) =$ ____________________ (a formula)

Shade the region whose area is $C(3) - C(1)$.

For each of the above, the **area** increases as x increases. So $A(x)$, $B(x)$ and $C(x)$ are increasing functions even though $f(x)$ is constant, $g(x)$ is increasing and $h(x)$ is decreasing. (There is a difficulty with $C(x)$ when x gets larger than 6. We'll deal with that later.)

1d Now calculate the derivatives of the area functions from problems 1, 2 and 3 above:

$A'(x) =$ ____________ $B'(x) =$ ____________ $C'(x) =$ ____________

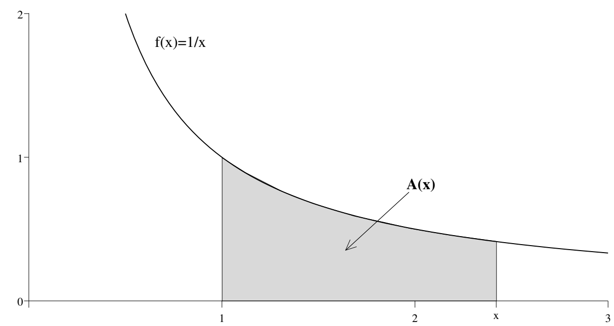
How is $A'(x)$ related to $f(x)$ in problem 1?

How is $B'(x)$ related to $g(x)$ in problem 2?

How is $C'(x)$ related to $h(x)$ in problem 3?

The Natural Logarithm

2a Define $A(x)$ to be the **area** bounded by the x-axis and the function $f(x) = 1/x$ between the line $x = 1$ and the vertical line at x. (See the diagram.)



Based on your work in problem 1,

$A'(x) =$ ________

Compute $A(1) =$ ________

Compute $A(x) =$ ________

2b So the area under $f(x) = 1/x$ between $x = 1$ and $x = 2$ is equal to $\ln(2)$. Outline this area on the graph. We'll use estimates of this area to compute approximations of $\ln(2)$.

2c Slice the area up into 4 pieces by drawing 3 evenly spaced vertical lines from the x-axis up to the curve.

2d Using the left side of each slice as the height, sketch in 4 rectangles on your graph. What are the x-coordinates of the sides of the rectangles? Plug these x-coordinates into $f(x) = 1/x$ to compute the heights of the rectangles. Find the areas of the 4 rectangles and add them up. This is your first approximation of the area under the curve, and $\ln(2)$. Is it an over-estimate or an under-estimate?

2e Using the right side of each slice as the height, sketch in 4 rectangles on your graph. Find the area of these rectangles and add them up. This is your second approximation of the area under the curve, and $\ln(2)$. Is it an over-estimate or an under-estimate?

2f Take the average of your two estimates to get a new estimate of $\ln(2)$. How does it compare with the value given by your calculator?

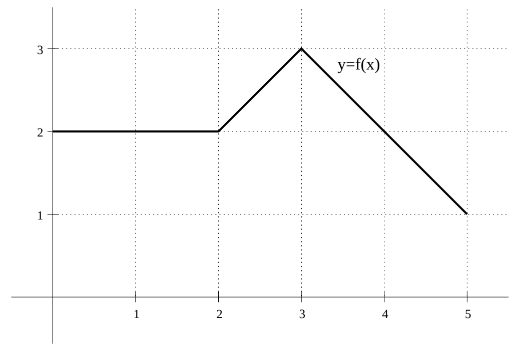
2g Use the midpoint of each slice to determine the height and sketch in the resulting 4 rectangles. Use them to approximate $\ln(2)$. Can you tell if you are getting an over-estimate or and under-estimate? Which of your four estimates gives you the closest answer to the value given by your calculator?

Name ________________________ Quiz Section ______

In this worksheet, we explore the Fundamental Theorem of Calculus and applications of the Area Problem to problems involving distance and velocity. We also consider integrals involving net and total change.

FTC Practice

1 Let $f(x)$ be given by the graph to the right and define $A(x) = \int_0^x f(t)\,dt$. Compute the following.



$A(1) =$____________ $A(2) =$____________

$A(3) =$____________ $A(4) =$____________

$A'(1) =$____________ $A'(2) =$____________

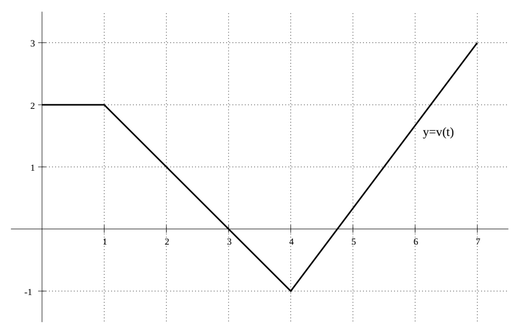
$A'(3) =$____________ $A'(4) =$____________

The maximum value of $A(x)$ on the interval $[0, 5]$ is ____________

The maximum value of $A'(x)$ on the interval $[0, 5]$ is ____________

Velocity and Distance

2 A toy car is travelling on a straight track. Its velocity $v(t)$, in m/sec, be given by the graph to the right. Define $s(t)$ to be the position of the car in meters. Choose coordinates so that $s(0) = 0$. Compute the following.



$s(2) =$__________ $s(4) =$__________ $s(6) =$__________

$v(2) =$__________ $v(4) =$__________ $v(6) =$__________

The maximum value of $s(t)$ on the interval $[0, 7]$ is ____________

The minimum value of $s(t)$ on the interval $[0, 7]$ is ____________

The maximum value of $v(t)$ on the interval $[0, 7]$ is ____________

The minimum value of $v(t)$ on the interval $[0, 7]$ is ____________

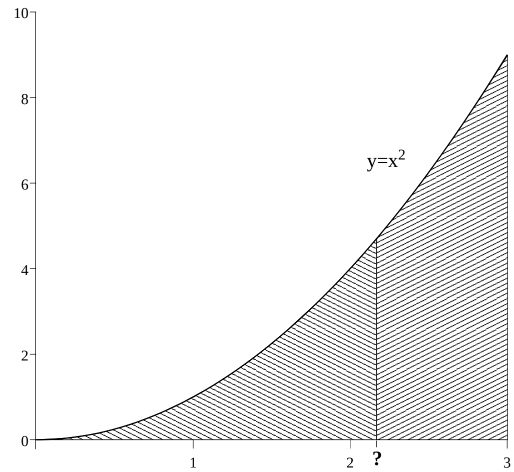
Net and Total Change

3 (a) Evaluate $\int_{-2}^{2} \left|x^2 - 4\right| \, dx$ and $\left|\int_{-2}^{2} \left(x^2 - 4\right) \, dx\right|$ and explain your answers.

(b) Now evaluate $\int_{-3}^{3} \left|x^2 - 4\right| \, dx$ and $\left|\int_{-3}^{3} \left(x^2 - 4\right) \, dx\right|$ and explain your answers.

Another Area Problem

4 An artist you know wants to make a figure consisting of the region between the curve $y = x^2$ and the x-axis for $0 \le x \le 3$



(i) Where should the artist divide the region with a vertical line so that each piece has the same area? (See the picture.)

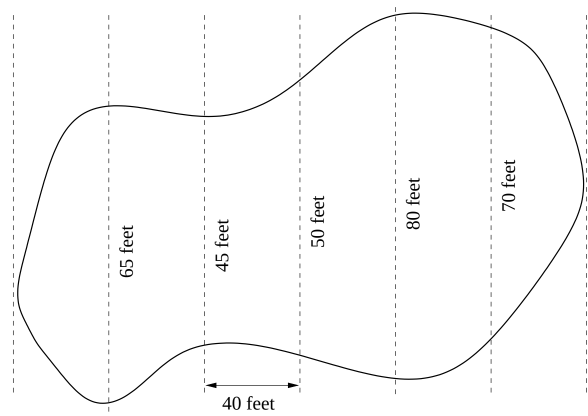
(ii) Where should the artist divide the region with vertical lines to get 3 pieces with equal areas?

Name ________________________ Quiz Section ______

In this work sheet we'll study the problem of finding the area of a region bounded by curves. We'll first estimate an area given numerical information. The we'll use calculus to find the area of a more complicated region.

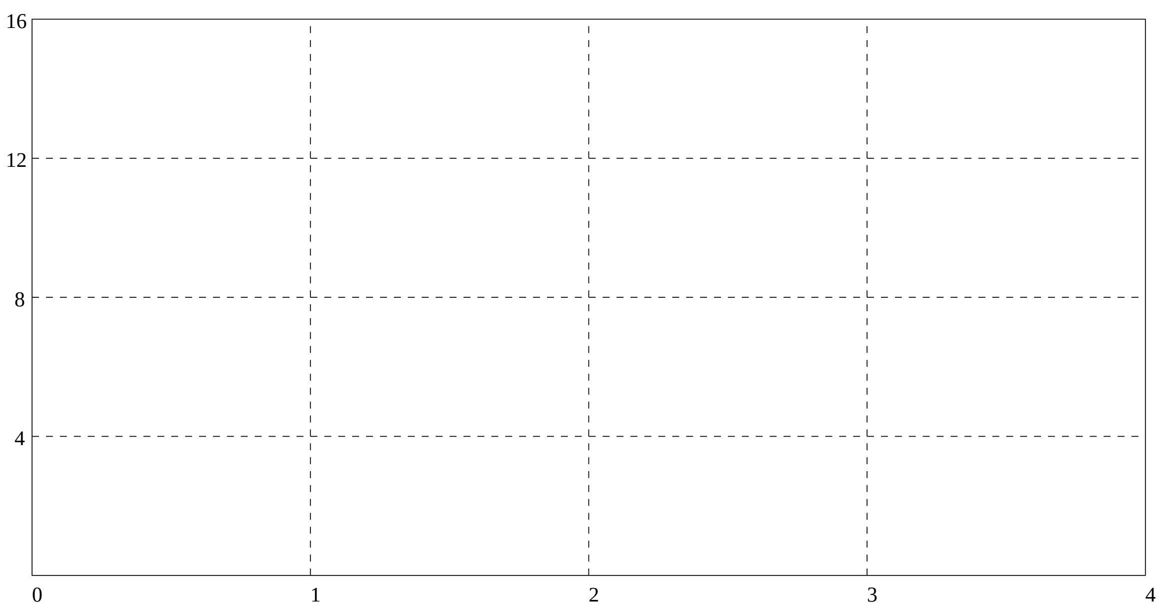
The Lake

1 The widths, in feet, of a small lake were measured at 40 foot intervals. Estimate the area of the lake.



Area Bounded by Three Curves

2 On the grid below sketch the graphs of $y = 4$, $y = x^2$ and $y = \sqrt{27x}$. (The last one is just a piece of a sideways parabola).



3 Shade the "triangular" region bounded by the graphs of the three functions that lies above the horizontal line.

4 Compute the x-coordinate of the left endpoint of the region.

5 Compute the x-coordinate of the right endpoint of the region.

6 Note that the top of the region consists of a single curve, but the bottom of the region consists of two different curves. Find the x-coordinate where these two curves meet.

7 Sketch in a vertical line at the x-coordinate you found in the last problem. This divides the region into two smaller sub-regions.

8 Compute the area of the left sub-region.

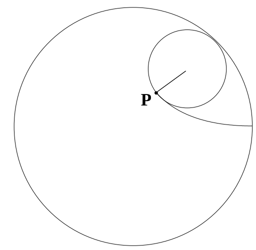
9 Compute the area of the right sub-region. Add the two areas together to get the total area.

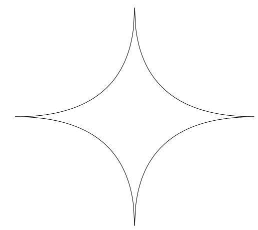
10 Recompute the area using the following trick. Solve for x as a function of y in the two non-constant functions. Find the area by integrating with respect to y. Is this easier?

Solids of Revolution and the Astroid — Math 125

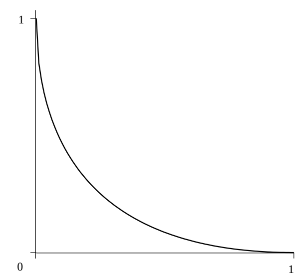
Name ________________________ Quiz Section ______

In this worksheet we are going to practice computing some more volumes of solids of revolution. These will all be based on a curve called the “astroid”. This curve is formed by rolling a small wheel around the inside of a larger one (see the picture). If the radius of the small wheel is one quarter the radius of the big one, a point P on the small wheel will trace out the four pointed curve shown on the far right. It’s called the astroid because it looks like a star.





1 If the radius of the big wheel is taken to be one, the astroid can be shown to have the equation $x^{2/3} + y^{2/3} = 1$. Use disks to compute the volume of the solid generated by rotating the part of the astroid in the first quadrant around the y-axis.



2 Use cylindrical shells to compute the volume of the solid generated by rotating the first quadrant portion of the astroid about the x-axis. (Hint: Try the substitution $u^3 = y^2$, so $3u^2\,du = 2y\,dy$.) How does this compare with your answer in Problem 1? Can you explain this geometrically?

3 Use any method you wish to compute the volumes of the solids generated by rotating the first quadrant portion of the astroid about the lines $x = 1$ and $y = -1$. **Set up only. Do not compute the integrals.**

Name ________________________ Quiz Section ______

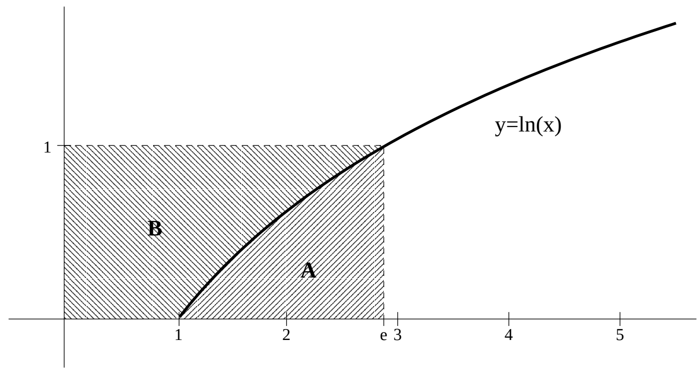
In this work sheet we'll study the technique of integration by parts. Recall that the basic formula looks like this:

$$\int u\,dv = u \cdot v - \int v\,du$$

1 First a warm-up problem. Consider the integral $\int x\,\sin(3x)\,dx$. Let $u = x$ and let $dv = \sin(3x)\,dx$. Compute du by differentiating and v by integrating, and use the basic formula to compute the original integral. Don't forget the arbitrary constant!

2 Compute $\int \ln x\,dx$. (The proper technique is, indeed, integration by parts. What should you take to be u and dv? The choices are pretty limited. Try one and see what happens.)

3 The regions A and B in the figure are revolved around the x-axis to form two solids of revolution.



(a) Before computing the integrals, which solid do you think has a larger volume? Why?

(b) Use the disk method to find the volume of the solid swept out by region A.

(c) Use the shell method to find the volume of the solid swept out by region B.

4 Suppose we try to integrate $1/x$ by parts, taking $u = 1/x$ and $dv = dx$. We have $du = (-1/x^2)\,dx$ and $v = x$, so

$$\begin{aligned}\int \frac{1}{x}\,dx &= \frac{1}{x}\cdot x - \int x \cdot \frac{-1}{x^2}\,dx \\ &= 1 + \int \frac{1}{x}\,dx.\end{aligned}$$

Canceling the integral from both sides, we get the disconcerting result that $0 = 1$. *What went wrong?* What happens if we replace the indefinite integrals by definite integrals, that is, if we try to calculate $\int_a^b \frac{1}{x}\,dx$ by this method?

Name ________________________ Quiz Section ______

Integration of rational functions is mostly a matter of algebraic manipulation. In this worksheet we shall work through some examples of the necessary techniques.

1a Consider the rational function $f(x) = \dfrac{2x^3 - 4x^2 - 5x + 3}{x^2 - 2x - 3}$. Use long division to get a quotient and a remainder, then write $f(x) =$quotient + (remainder/divisor).

1b Now consider the expression $\dfrac{x+3}{x^2 - 2x - 3}$. Factor the denominator into two linear terms.

1c We wish to write $\dfrac{x+3}{(x-3)(x+1)}$ as a sum $\dfrac{A}{x-3}+\dfrac{B}{x+1}$. Let's find A and B. Set the two expressions equal and clear denominators (that is, multiply through by $(x-3)(x+1)$ and cancel $(x-3)$'s and $(x+1)$'s as much as possible). Plug in $x=3$ and solve for A. Use the same idea to find B. Check your work by adding the two fractions together.

1d Now use the results of Problems 1a, 1b, and 1c to compute $\displaystyle\int \frac{2x^3-4x^2-5x+3}{x^2-2x-3}\,dx$.

1e Some of the terms in the answer to Problem 1d involve logarithms. Combine those terms into a single term of the form ln(some function of x).

2a Next, consider the function $f(x) = \frac{3x+1}{x(x+1)^2}$. The problem here is that one of the linear factors in the denominator is squared. Partial fraction theory says the best we can do is to get this one in the form $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$. Let's find A, B and C.

First set the two expressions equal and clear denominators. Plug in $x = 0$ and solve for A. Plug in $x = -1$ and solve for C.

2b Now that you've found A and C, you can find B by plugging in any other convenient value for x. Do so.

2c Now compute $\int \frac{3x+1}{x(x+1)^2}\,dx$.

Techniques of Integration Math 125

Name ________________________ Quiz Section ______

The following integrals are more challenging than the basic ones we've seen in the textbook so far. You will probably have to use more than one technique to solve them. Don't hesitate to ask for hints if you get stuck.

1. $$\int \frac{\sin(t)\cos(t)}{\sin^2(t) + 6\sin(t) + 8}\, dt$$

2. $$\int \left(\sin^{-1}(x)\right)^2\, dx$$

3. $\displaystyle\int \frac{y^2}{(1-y^2)^{7/2}}\,dy$

4. $\displaystyle\int \frac{1}{x+2\sqrt{x}+1}\,dx$

Name ________________________ Quiz Section ______

This worksheet helps you review improper integrals and approximation techniques.

Improper Integrals

1 Show that $\int_1^\infty \frac{1-\sin x}{x^2}\,dx$ converges, using the Comparison Test.

2 Use Problem 1 and the fact that $\int_1^\infty \frac{1}{x^2}\,dx$ converges to show that $\int_1^\infty \frac{\sin x}{x^2}\,dx$ converges. Why can't we use the Comparison Test directly on this one?

3 Use Problem 2 and integration-by-parts to show that $\int_1^\infty \frac{\cos x}{x}\,dx$ converges.

Approximation Techniques

Let $f(x) = \dfrac{\ln(x)}{x}$. We will use geometric reasoning to see if the Trapezoid Rule gives an overestimate, or an underestimate of the integral $\displaystyle\int_1^3 f(x)\,dx$.

1 Compute $f'(x)$. Where is the function increasing and decreasing? (We only care about the interval $1 \leq x \leq 3$.)

2 Compute $f''(x)$. Is the function concave up or concave down on the interval?

3 Sketch a graph of $y = f(x)$ on the interval $1 \leq x \leq 3$. Sketch the approximate area under the curve given by the Trapezoid Rule with $n = 4$ subintervals. Is it an overestimate or an underestimate? How is the answer related to Problem 2?

Name ________________________ Quiz Section ______

This worksheet walks you through a couple of non-trivial applications of Differential Equations.

Forensic Mathematics

A detective discovers a murder victim in a hotel room at 9:00am one morning. The temperature of the body is 80.0°F. One hour later, at 10:00am, the body has cooled to 75.0°F. The room is kept at a constant temperature of 70.0°F. Assume that the victim had a normal temperature of 98.6°F at the time of death. We'll use differential equations to find the time the murder took place.

Let $u(t)$ be the temperature of the body after t hours. By Newton's Law of Cooling we have the differential equation

$$\frac{du}{dt} = k\,(u - 70)$$

where k is a constant (to be determined). We'll solve the differential equation and get a formula for $u(t)$.

1 Multiply both sides by dt to get a differential form of the equation. Now do some easy algebra to get the variable u on the same side as the du. (Leave the k where it is).

2 Integrate both sides of the equation. Integrate the right side with respect to t and the left with respect to u. You can combine the integration constants into one "$+C$" on the right side.

3 Solve for u as a function of t. Your function will involve the constants k and C.

4 Take $t = 0$ when the body was found at 9:00am. Plug in $t = 0$ and $u = 80.0°\text{F}$ and solve for C. (It's easier to solve for $A = e^C$ and use this in your formula).

5 Plug in $t = 1$ and $u = 75.0°\text{F}$ and solve for k. (This'll take some log tricks).

6 Set $u = 98.6°\text{F}$ and solve for t. At what time did the murder take place?

Spread of a Rumor

The Xylocom Company has 1000 employees. On Monday a rumor began to spread among them that the CEO had suddenly moved to Brazil. It is reasonable to assume that the rate of the spread of the rumor is proportional to the number of possible encounters between employees who have heard the rumor and those who have not. Let $y = y(t)$ be number of employees who have heard the rumor after t days.

1 Explain why the number of possible meetings between employees who have heard the rumor and those who have not equals $y\,(1000 - y)$.

2 Write a differential equation that describes this model of the spread of a rumor. (Remember that "is proportional to" means "is some constant k times".)

3 Proceed as in the cooling body problem to solve the differential equation for $y(t)$. You will need to use the method of partial fractions. Your answer should involve two constants: the proportionality constant k and a constant C from integrating. (As in part 2 of the previous problem, you can combine the integration constants into one "$+C$" on the right side.)

4 At the very beginning, 50 people had heard the rumor (they all attended the same meeting). Compute the constant $A = e^C$.

5 On Tuesday morning, 100 people had heard the rumor. Compute the constant k.

6 When will 800 people have heard the rumor?

Name ________________________ Quiz Section ______

The following problems should help you review for the final exam. Don't hesitate to ask for hints if you get stuck.

Integration Techniques

1. Evaluate the following integrals.

(a) $\int_0^{\pi/2} \cos^5(x)\,dx$

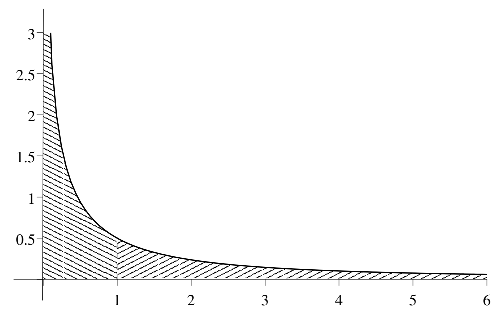
(b) $\int \frac{\sqrt{4-9x^2}}{x}\,dx$

(c) $\int_1^e x^3 \ln(x)\,dx$

(d) $\int \frac{\sqrt{x+3}}{x+2}\,dx$

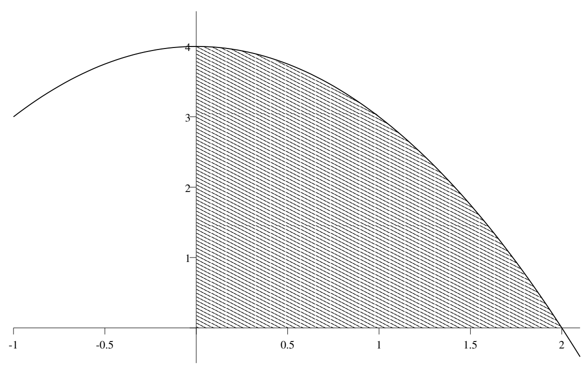
Improper Integrals

2. Compute the integral $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$. (Hint: write it as the sum of two integrals.)



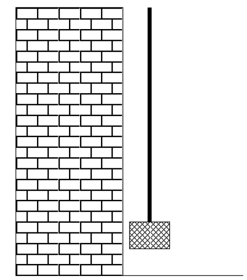
Volumes

3. The region to the right of the y-axis, above the x-axis, and below the curve $y = 4 - x^2$ is revolved about the y-axis. Set up an integral that represents the volume of the resulting solid
(a) by the method of slices (discs)
(b) by the method of cylindrical shells.
Then find the volume, by either method you like.



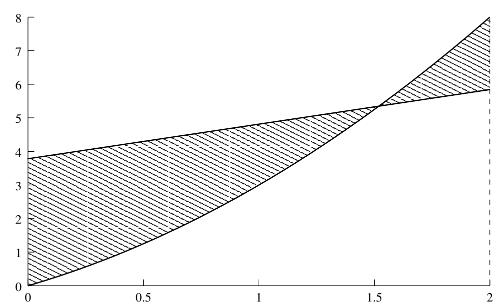
Work

4. A cable hanging from the top of a building is 15m long and has a mass of 40kg. A 10kg object is attached to the end of the rope. How much work is required to pull 5m of the cable up to the top?



Area Between Curves

5. Find the area of the region bounded by $y = x^2 + 2x$ and $y = x + 3.75$ between $x = 0$ and $x = 2$.



Name ________________________ Quiz Section ______

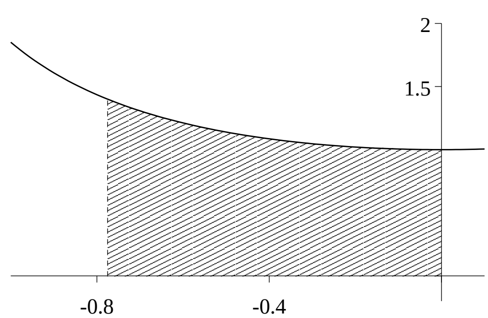
The following problems should help you review for the final exam. Don't hesitate to ask for hints if you get stuck.

Arclength and Approximation

1. Write an integral that computes the arclength of the curve $y = e^{x/2}$ between $x = 0$ and $x = 2$. Use Simpson's Rule with $n = 4$ subintervals to estimate the value of the integral.

Center of Mass

2. Find the center of mass of a plate with constant density that occupies the region $-\frac{1}{4}\pi \leq x \leq 0, \quad 0 \leq y \leq \sec^2 x$.



Net and Total Distance

3. You throw a ball straight up into the air with velocity 40 ft/sec and catch it (at the same height) when it comes back down. What is the total distance traveled by the ball?

Differential Equations

4. Let $f(t)$ be a continuous function and let a be a constant. Show that $y = e^{-at} \int_0^t e^{as} f(s)\, ds$ satisfies the differential equation $\frac{dy}{dt} + ay = f(t)$.

5. An electric circuit with resistance 10 ohms and inductance 2 henrys is powered by a 12-volt battery. The current I (in amperes) at time t (in seconds) in such a circuit satisfies the differential equation

$$2\frac{dI}{dt} + 10I = 12.$$

Suppose that $I = 0$ when the circuit is activated at time $t = 0$.
(a) Find the current I at all times $t > 0$.
(b) Find the limiting value of I as $t \to \infty$.
(c) After what time is the current within 0.1 ampere of its limiting value?