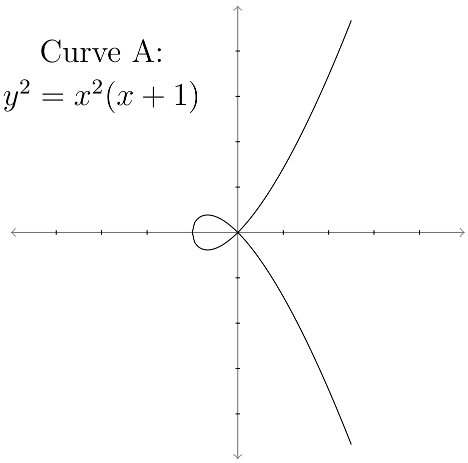
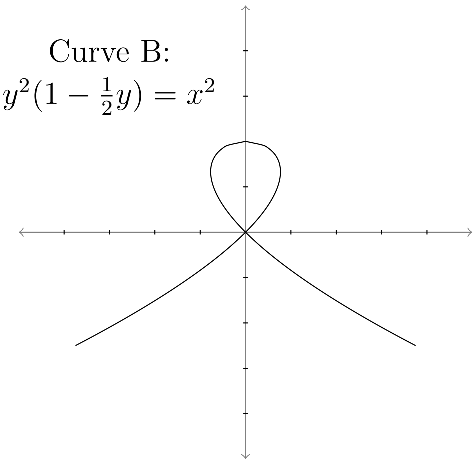
Worksheet for Week 6: Implicit Differentiation

In this worksheet, you'll use parametrization to deal with curves that have more than one tangent line at a point. Then you'll use implicit differentiation to relate two derivative functions, and solve for one using given information about the other.





1. (a) Use impicit differentiation to find all the points in Curve A with a horizontal tangent line. (Looking at the graph, how many such points should there be?)

> **Solution:** Using implicit differentiation, we get $\frac{dy}{dx} = \frac{x(3x+2)}{2y}$, so the points where $dy/dx = 0$ are those with x-coordinate $-\frac{2}{3}$. (We can't set $x = 0$, because then the equation for Curve A says that y is also 0. Then dy/dx has a zero in the denominator.) Solving for y, we get the two points $\left(-\frac{2}{3}, \frac{2}{3\sqrt{3}}\right)$ and $\left(-\frac{2}{3}, -\frac{2}{3\sqrt{3}}\right)$. ($\frac{2}{3\sqrt{3}} \approx 0.3849....$)

(b) What about Curve B?

> **Solution:** Using implicit differentiation, we get
>
> $$\frac{dy}{dx} = \frac{2x}{2y - \frac{3}{2}y^2},$$
>
> so the points where $dy/dx = 0$ are those with $x = 0$ and y nonzero. If $x = 0$ on the curve, then either $y = 0$ or $y = 2$, so we have only the point $(0, 2)$.

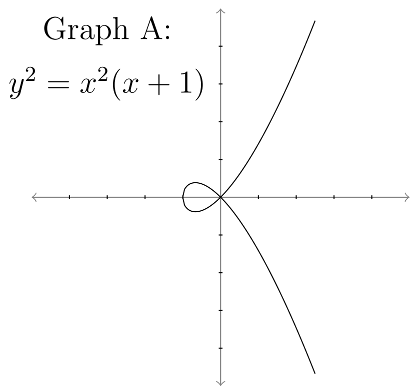
(c) Try to find $\frac{dy}{dx}$ at the point $(0, 0)$ on both graphs. What goes wrong?

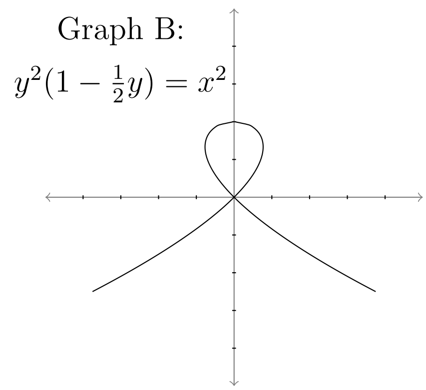
> **Solution:** It's not possible to plug in $x = 0$ and $y = 0$ to either of the expressions for $\frac{dy}{dx}$. The derivative function tells us nothing about $(0, 0)$.

2. In this problem you'll look at the curves from Page 1 in a different way.

Suppose a cat is chasing a ball around on the floor, and its position is described by the parametric equations

$$(x(t), y(t)) = (t^2 - 1, t - t^3).$$





(a) The cat is following one of the paths from the previous page (reprinted above). Which path does the cat follow? Circle this curve. How do you know it's the right one?

> **Solution:** The expressions for $x(t)$ and $y(t)$ satisfy the equation corresponding to Curve A. That is, $y(t)^2 = x(t)^2(x(t) + 1)$.

(b) Draw an arrow on the circled graph above, to indicate in which direction the cat is running.

> **Solution:** As long as t is bigger than 1, the expression $y(t) = t - t^3$ is negative, so the cat eventually stays in the lower half of the plane. The cat enters the picture in the upper right, runs around a loop near the origin, and runs away to the lower right.

(c) At which time(s) t does the cat run through the point $(0, 0)$?

Solution: If $x(t) = 0$, then $t = \pm 1$, and if $y = 0$ then $t = \pm 1$ or $t = 0$. So the only times when x and y are both zero are $t = \pm 1$.

(d) Remember that it wasn't possible to find $\frac{dy}{dx}$ at $(0, 0)$ using the method on Page 1. But now that the graph has been parametrized, you can do it. What are the tangent line(s) to the parametrized curve $(x(t), y(t))$ at $(0, 0)$?

Solution: We'll use the chain rule, which says that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Now

$$\frac{dy}{dt} = 1 - 3t^2, \qquad \frac{dx}{dt} = 2t.$$

At $t = 1$ we have $\frac{dy}{dt} = -2$ and $\frac{dx}{dt} = 2$; at $t = -1$ we have $\frac{dy}{dt} = -2$ and $\frac{dx}{dt} = -2$. So the slopes are -1 and 1, respectively, and the equations of the tangent lines are

$$y = -x \qquad \text{and} \qquad y = x.$$

3. This next question is a new type of problem that you can solve now that you know about implicit differentiation. Suppose a snowball is rolling down a hill, and its radius r is growing at a rate of 1 inch per minute. The volume V of the snowball grows more quickly as the snowball gets bigger. In this question, you'll find the rate of change of the volume, $\frac{dV}{dt}$, at the instant when the radius r is 6 inches.

   (a) First, apply geometry to the situation. Can you think of an equation that relates the variables r and V to each other?

   Solution: $V = \frac{4}{3}\pi r^3$.

   (b) Now the variables V and r change as time changes, so we can think of them as functions of t. Differentiate the equation you came up with in part (a) with respect to t.

   Solution: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

   (c) What is the rate of change of the radius? Use this to simplify your equation from part (b).

   Solution: The rate is given; $\frac{dr}{dt} = 1$. (Units: inches/minute.)

   (d) What is the rate of change of V when the radius of the snowball is 6 inches?

   Solution: $\frac{dV}{dt} = 4\pi(6)^2 = 144\pi$. (Units: cubic inches/minute.)