

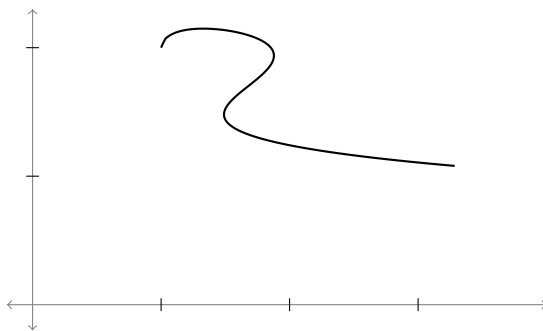
Worksheet for Week 4: Velocity and parametric curves

In this worksheet, you'll use differentiation rules to find the vertical and horizontal velocities of an object as it follows a parametric curve. You'll also get a preview of how to find tangent lines to parametric curves.

1. A bug is running around on a window. If we put x and y axes on the window, then the bug follows the path given by the parametric equations

$$(x(t), y(t)) = \left(\frac{1}{20}te^t + e^t - t^e, t^{2/3} - t + 2 \right).$$

The bug's path is shown below, for $0 \leq t \leq 3$.



Use the equations for the bug's path to answer the following questions.

- (a) The horizontal velocity of the bug at some time t is given by the derivative $x'(t)$. What is a formula for $x'(t)$?

Solution: Use the product rule and the rules for differentiating polynomials and exponential functions. We'll do it term by term below:

$$\begin{aligned} \frac{d}{dt} \left(\frac{t}{20} e^t \right) &= \frac{t}{20} \cdot \frac{d}{dt}(e^t) + \frac{d}{dt} \left(\frac{t}{20} \right) \cdot e^t \quad (\text{product rule}) \\ &= \frac{t}{20} e^t + \frac{1}{20} e^t \\ &= \frac{e^t}{20} (t + 1) \end{aligned}$$

$$\frac{d}{dt}(e^t) = e^t \quad (\text{exponential rule})$$

$$\frac{d}{dt}(-t^e) = -et^{e-1} \quad (\text{polynomial rule})$$

$$\text{So } x'(t) = \frac{e^t}{20}(t + 1) + e^t - et^{e-1}.$$

- (b) Now find a formula for $y'(t)$, the vertical velocity of the bug.

Solution: Here we just need the differentiation rules for powers of x .

$$\frac{d}{dt} (t^{2/3} - t + 2) = \frac{2}{3}t^{-1/3} - 1.$$

- (c) Find a point on the path where the vertical velocity y' is zero.

Solution: First, set the formula from Part (b) equal to 0 to find a time when the vertical velocity is 0:

$$\begin{aligned}\frac{2}{3}t^{-1/3} - 1 &= 0 \\ \frac{2}{3} &= t^{1/3} \\ \frac{8}{27} &= t.\end{aligned}$$

Now plug that time t into the parametric equations to find the point:

$$\left(x\left(\frac{8}{27}\right), y\left(\frac{8}{27}\right) \right) \approx (1.33, 2.15)$$

- (d) Now look back at the picture of the bug's path on the first page. What can you say about the tangent line to the path at the point you found in Part (c)? Explain why your answer makes sense, given what you know about y' there.

Solution: The tangent line appears to be horizontal.

Let's think about why this makes sense. If we plug the t -value from Part (c) into the formula for $x'(t)$, we see that x' is positive there: the bug is moving to the right at that instant. Also, the bug is not moving up or down at all: the rate of change of its y -position is 0. So at that instant, the bug is *only* moving to the right. Thus the tangent line should be horizontal.

- (e) Suppose (a, b) is a point where the horizontal velocity x' is zero, and the vertical velocity y' is *not* zero. What do you predict the tangent line looks like at this point? Why?

Solution: Similarly, if the horizontal velocity is zero at some time t , where $(x(t), y(t)) = (a, b)$, then the bug is not moving to the left or right at that instant. Also, since the vertical velocity is nonzero, we know that the bug is moving either up or down. Thus we have upward/downward motion, but no right/left motion. So the tangent line to the path at (a, b) should be vertical.

2. Now suppose a spider is following the path given by the parametric equations

$$(x(t), y(t)) = (t - 2, t^{4/3} - 2t).$$

- (a) At which point (a, b) is the vertical velocity of the spider equal to 0?

Solution: First, find $y'(t)$:

$$\frac{d}{dt}(t^{4/3} - 2t) = \frac{4}{3}t^{1/3} - 2.$$

Now set it equal to 0, and solve:

$$t = \frac{27}{8}.$$

Plug this into the equations for x and y :

$$(x, y) \approx (1.36, -1.69).$$

- (b) If a curve is given parametrically and if t_0 is a time where $x'(t_0) \neq 0$, then the slope of the tangent line to the curve at the point $(x(t_0), y(t_0))$ is given by

$$\frac{y'(t_0)}{x'(t_0)}.$$

(You will see this again in class.)

Using this information, find the point (a, b) on the spider's path where the tangent line is perpendicular to the line $y = -5x - 12$.

Solution: We need to find a t -value so that the slope of the tangent line to the path at $(x(t), y(t))$ is $\frac{1}{5}$. Using the fact given above, the formula for the slope of the tangent line at time t , for $x'(t) \neq 0$, is

$$\frac{x'(t)}{y'(t)} = \frac{\frac{4}{3}t^{1/3} - 2}{1} = \frac{4}{3}t^{1/3} - 2.$$

Set it equal to $\frac{1}{5}$, and solve for t :

$$t = \left(\frac{33}{20}\right)^3 \approx 4.49.$$

Plug this t -value into the parametric equations for x and y :

$$(x, y) \approx (2.49, -1.57).$$