Worksheet for Week 9: Indeterminate Forms and L'Hospital's Rule

We worked with limits at the beginning of the quarter, sometimes using algebraic tricks to compute them. Now, we return to limits which have *indeterminate forms* and use L'Hospital's Rule to evaluate them. Although we have seven indeterminate forms

$$\frac{"\infty"}{\infty},\quad \frac{"0"}{0},\quad "\infty-\infty","\infty\cdot 0",\quad "1^{\infty}",\quad "0^{0}",\quad "\infty^{0}"$$

(all in quotes because they are not actual mathematical expressions to be evaluated), L'Hospital's Rule only works with indeterminate quotients.

L'Hospital's Rule

Suppose f and g are differentiable functions and $g'(x)$ is not identically zero near a, except possibly at $x = a$. Suppose that

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty.$$

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

if the limit on the right exists (or is ∞ or $-\infty$).

The rule is actually more general and it works with limits $x \to \infty$, $x \to -\infty$, $x \to a^-$ and $x \to a^+$, as well.

It is important that you have an $\frac{"\infty"}{\infty}$ or $\frac{"0"}{0}$ indeterminate form before you use L'Hospital's Rule.

1. **Indeterminate Quotients and L'Hospital's Rule**

   For each of the limits below, determine if it is the indeterminate quotient $\frac{"\infty"}{\infty}$, the indeterminate quotient $\frac{"0"}{0}$, or if it is not indeterminate. If you have an indeterminate quotient evaluate the limit using L'Hospital's Rule. If the limit is not indeterminate, you should be able to evaluate it without much effort.

   (a) $\lim\limits_{x\to-\infty} \dfrac{1+\sqrt{3-x}}{x}$

   (b) $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3} =$

   (c) $\lim\limits_{x\to 1} \dfrac{x^3 - x}{x^3 - 2x^2 - 3x} =$

(d) $\lim_{x \to 2} \frac{x^2 + x - 6}{x^3 - 4x^2 + 4x}$

(e) $\lim_{x \to \infty} \frac{\ln x}{x^a}$ (*Be careful. Your answer will depend on a.*)

(f) $\lim_{x \to 0} \frac{x - \sin^{-1} x}{x^3}$

2. **L'Hospital may not always be the best way to go**
   The following limits are all indeterminate quotients. Although L'Hospital works on most (but not all!) of them, it may not be the best approach. Some of the limits, try to remember older tricks from the second week of the course for evaluating limits. For others, you may want to split up and work in pieces.

   (a) $\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{3+2x}$

   (b) $\lim_{x\to\infty} \frac{x+\cos x}{x+\sin x}$

(c) $\lim_{x \to 0} \frac{\sin^4(x)\sin^6(3x)}{5x^{10}}$

(d) $\lim_{x \to \infty} \frac{5x^2 3^{1+x} + 3x^4 \ln(x)}{2x^2 3^{3+x} + 9x^4 \ln(x)}$

3. **Indeterminate difference "$\infty - \infty$" and indeterminate product "$0 \cdot \infty$"**
Determine if the following limits are one of the indeterminate forms "$\infty - \infty$" or "$0 \cdot \infty$". If yes, state which one and use algebra to turn it into an indeterminate quotient to evaluate using L'Hospital Rule or some other idea. If the limit is not indeterminate, you should be able to evaluate it without much effort.

(a) $\lim_{u \to \infty} u \sin(1/u)$

(b) $\lim_{x \to \infty} \left(\sqrt{16x^2 + 4x} - 4x \right)$

(c) $\lim_{t \to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right)$

(d) $\lim_{x \to -\infty} \left(\sqrt{25x^2 + x} - 5x \right)$

4. **Indeterminate Powers "1^∞", "0^0", and "∞^0"**
Determine if the following limits are one of the indeterminate powers "1^∞", "0^0", or "∞^0". If yes, state which indeterminate power and evaluate by first using the natural logarithm function to bring the power down. (Which new indeterminate form do you have after taking the logarithm of both sides?)

(a) $\lim_{x\to 0^+} x^{\frac{1}{x}}$

(b) $\lim_{x\to 0^+} (\sin x)^x$

(c) $\lim_{x\to\infty} \left(\frac{x+b}{x-1}\right)^{x-1}$