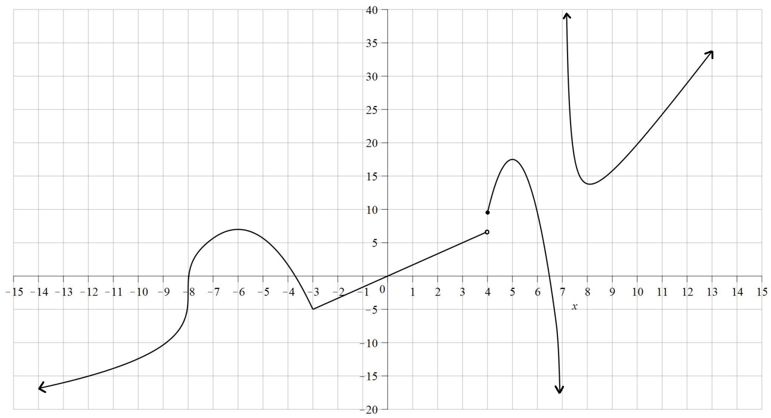
Worksheet for Week 4: Limits and Derivatives

This worksheet reviews limits and the definition of the derivative with graphs and computations.

1. Answer the following questions using the graph $y = f(x)$ below. The function $f(x)$ has domain all numbers except 7 as seen from the graph.



(a) $\lim_{x\to 4} f(x) = \text{DNE}$

(b) $\lim_{x\to 7^+} f(x) = \infty$

(c) $f'(0) = \frac{5}{3}$

(d) $\lim_{x\to -3} f(x) = -5$

(e) $\lim_{x\to 0} \dfrac{f(x)}{x} = f'(0) = \frac{5}{3}$

(f) $\lim_{h\to 0} \dfrac{f(3+h)-5}{h} = f'(3) = \frac{5}{3}$

(g) $f'(5) = 0$

(h) $\lim_{h\to 0^+} \dfrac{f(-8+h)-f(-8)}{h} = \infty$

(i) $\lim_{h\to 0} \dfrac{f(-8+h)}{h} = \infty$

(j) $\lim_{h\to 0} \dfrac{f(-6+h)-f(-6)}{h} = f'(-6) = 0$

(k) $\lim_{h\to 0^+} \dfrac{f(-3+h)+5}{h} = \frac{5}{3}$

(l) List all the intervals where the derivative $f'(x)$ is negative. $(-6,-3), (5,7), (7,8)$

(m) List all the intervals where the derivative $f'(x)$ is decreasing. $(-8,-3), (4,7)$

(n) A critical value for $f(x)$ is any x in the domain of $f(x)$ where $f'(x) = 0$ or $f'(x)$ is undefined. List all critical values of $f(x)$. $x = -8, -6, -3, 4, 5, 8$

2. Evaluate the following limits and then match the functions with their graphs shown below using your limit results. Some will require you to compute left and right hand limits.

(a) $\lim_{x\to 5} \frac{1}{x-5} =$

Solution:

$$\lim_{x\to 5^+} \frac{1}{x-5} = \infty \quad \text{and} \quad \lim_{x\to 5^-} \frac{1}{x-5} = -\infty$$

so the limit Does Not Exist.

(b) $\lim_{x\to 5} \frac{-x}{(x-5)^2} =$

Solution:

$$\lim_{x\to 5} \frac{-x}{(x-5)^2} = -\infty$$

(c) $\lim_{x\to 5} \frac{-x^2-2x+35}{x^2-4x-5} =$

Solution:

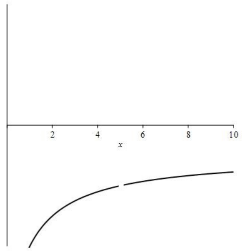
$$\lim_{x\to 5} \frac{-x^2-2x+35}{x^2-4x-5} = \lim_{x\to 5} \frac{-(x-5)(x+7)}{(x-5)(x+1)} = \lim_{x\to 5} \frac{-(x+7)}{x+1} = \frac{-12}{6} = -2$$

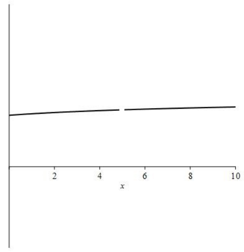
(d) $\lim_{x\to 5} \frac{x-\sqrt{3x+10}}{x-5} =$

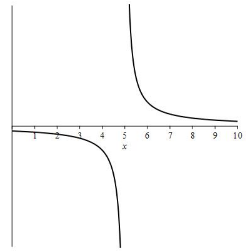
Solution:

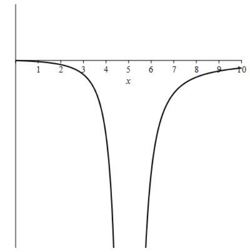
$$\lim_{x\to 5} \frac{x-\sqrt{3x+10}}{x-5} \cdot \frac{x+\sqrt{3x+10}}{x+\sqrt{3x+10}} = \lim_{x\to 5} \frac{x^2-3x-10}{(x-5)\left(x+\sqrt{3x+10}\right)}$$

$$= \lim_{x\to 5} \frac{(x-5)(x+2)}{(x-5)\left(x+\sqrt{3x+10}\right)} = \lim_{x\to 5} \frac{x+2}{x+\sqrt{3x+10}} = \frac{7}{10}$$









$$y = \frac{-x^2 - 2x + 35}{x^2 - 4x - 5} \qquad y = \frac{x - \sqrt{3x + 10}}{x - 5} \qquad y = \frac{1}{x - 5} \qquad y = \frac{-x}{(x - 5)^2}$$

3. Use the definition of the derivative $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ to compute $f'(3)$ for the following functions. Then match the functions with their graphs shown below using your limit results.

(a) $f(x) = (x-3)^{\frac{1}{3}} + 2$

Solution:

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h-3)^{1/3} + 2 - 2}{h} = \lim_{h\to 0} \frac{h^{1/3}}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty$$

so $f'(3)$ does not exist.

(b) $f(x) = (x-3)^{\frac{2}{3}} + 2$

Solution:

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h-3)^{2/3} + 2 - 2}{h} = \lim_{h\to 0} \frac{h^{2/3}}{h} = \lim_{h\to 0} \frac{1}{h^{1/3}}$$

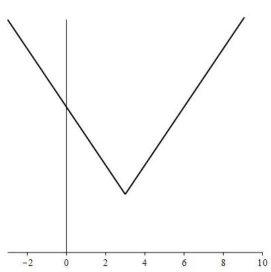
The limit does not exists because the left and right limits are $-\infty$ and ∞, respectively. So $f'(3)$ does not exist.

(c) $f(x) = |x-3| + 2$

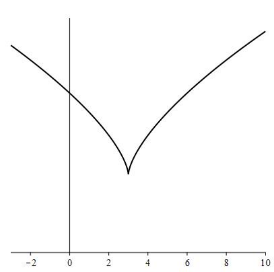
Solution:

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{|3+h-3| + 2 - 2}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

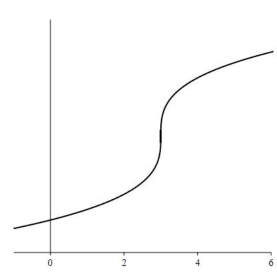
The limit does not exists because the left and right limits are -1 and 1, respectively. So $f'(3)$ does not exist.



$y = |x-3| + 2$



$y = (x-3)^{\frac{2}{3}} + 2$



$y = (x-3)^{\frac{1}{3}} + 2$

4. Find, if any, the horizontal asymptotes of the following functions and use that information to match them with their graphs on the next page. Each question should have two limit computations with $x \to \infty$ and $x \to -\infty$.

   (a) $f(x) = \dfrac{(x+1)^4}{x^4+3x^2+7x+10}$

   Solution:

   $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(x+1)^4}{x^4+3x^2+7x+10} = \lim_{x\to\infty} \frac{\frac{(x+1)^4}{x^4}}{\frac{x^4+3x^2+7x+10}{x^4}}$$

   $$= \lim_{x\to\infty} \frac{\left(\frac{x+4}{x}\right)^4}{\frac{x^4}{x^4}+\frac{3x^2}{x^4}+\frac{7x}{x^4}+\frac{10}{x^4}} = \lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)^4}{1+\frac{3}{x^2}+\frac{7}{x^3}+\frac{10}{x^4}} = \frac{(1+0)^4}{1+0+0+0} = 1$$

   $\lim\limits_{x\to-\infty} f(x)$ has the same steps and answer so $y=1$ is the horizontal asymptote on both sides.

   (b) $f(x) = \dfrac{x+3}{x^2+8x+26}$

   Solution:

   $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+3}{x^2+8x+26} = \lim_{x\to\infty} \frac{\frac{x+3}{x^2}}{\frac{x^2+8x+26}{x^2}}$$

   $$= \lim_{x\to\infty} \frac{\frac{x}{x^2}+\frac{3}{x^2}}{\frac{x^2}{x^2}+\frac{8x}{x^2}+\frac{26}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{3}{x^2}}{1+\frac{8}{x}+\frac{26}{x^2}} = \frac{0+0}{1+0+0} = 0$$

   $\lim\limits_{x\to-\infty} f(x)$ has the same steps and answer so $y=0$ is the horizontal asymptote on both sides.

   (c) $f(x) = \dfrac{x^3+4x+9}{x^2+4}$

   Solution:

   $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3+4x+9}{x^2+4} = \lim_{x\to\infty} \frac{\frac{x^3+4x+9}{x^3}}{\frac{x^2+4}{x^3}}$$

   $$= \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{4x}{x^3}+\frac{9}{x^3}}{\frac{x^2}{x^3}+\frac{4}{x^3}} = \lim_{x\to\infty} \frac{1+\frac{4}{x^2}+\frac{9}{x^3}}{\frac{1}{x}+\frac{4}{x^3}} = \infty$$

   The other limit has similar steps:

   $$\lim_{x\to-\infty} \frac{x^3+4x+9}{x^2+4} = ... = \lim_{x\to-\infty} \frac{1+\frac{4}{x^2}+\frac{9}{x^3}}{\frac{1}{x}+\frac{4}{x^3}} = -\infty$$

   because the denominator now is taking negative values with $x \to -\infty$. The graph of this function has no horizontal asymptotes.

(d) $f(x) = -7x^4 + x^3 - 12x + 20$

Solution:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} -7x^4 + x^3 - 12x + 20 = -\infty$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -7x^4 + x^3 - 12x + 20 = -\infty$$

The graph of this function has no horizontal asymptotes.

(e) $f(x) = \dfrac{\sqrt{8x^2+4}}{x+2}$

Solution:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{8x^2+4}}{x+2} = \lim_{x\to\infty} \frac{\frac{\sqrt{8x^2+4}}{x}}{\frac{x+2}{x}} = \lim_{x\to\infty} \frac{\frac{\sqrt{8x^2+4}}{\sqrt{x^2}}}{\frac{x+2}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{8x^2+4}{x^2}}}{\frac{x+2}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{8x^2}{x^2}+\frac{4}{x^2}}}{\frac{x}{x}+\frac{8}{x}} = \lim_{x\to\infty} \frac{\sqrt{8+\frac{4}{x^2}}}{1+\frac{8}{x}} = \sqrt{8}$$

Now for the next one look carefully to see where the difference in steps is:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\sqrt{8x^2+4}}{x+2} = \lim_{x\to-\infty} \frac{\frac{\sqrt{8x^2+4}}{x}}{\frac{x+2}{x}} = \lim_{x\to-\infty} \frac{\frac{\sqrt{8x^2+4}}{-\sqrt{x^2}}}{\frac{x+2}{x}}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{\frac{8x^2+4}{x^2}}}{\frac{x+2}{x}} = -\lim_{x\to-\infty} \frac{\sqrt{\frac{8x^2}{x^2}+\frac{4}{x^2}}}{\frac{x}{x}+\frac{8}{x}} = -\lim_{x\to-\infty} \frac{\sqrt{8+\frac{4}{x^2}}}{1+\frac{8}{x}} = -\sqrt{8}$$

When you use $\sqrt{x^2} = x$ you have to be careful because it is ONLY true when $x \geq 0$! If $x < 0$ (in this case $x \to -\infty$) we have $\sqrt{x^2} = -x$. Try $x = -3$, for example.

So this function has two horizontal asymptotes: The graph approaches $y = \sqrt{8}$ on the right as $x \to \infty$ and it approaches $y = -\sqrt{8}$ on the left as $x \to -\infty$.

(f) $f(x) = 3e^x$

Solution:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 3e^x = \infty$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 3e^x = 0$$

The graph has $y = 0$ as a horizontal asymptote on the left side only.

(g) $f(x) = 7 - e^{-x}$

Solution:

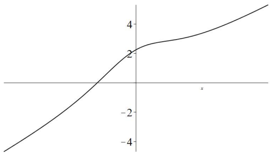
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 7 - e^{-x} = 7 - 0 = 7$$

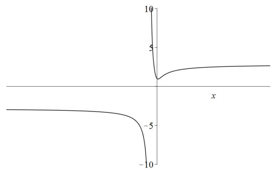
and

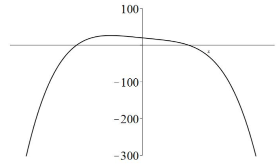
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 7 - e^{-x} = -\infty$$

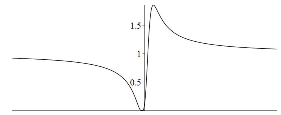
The graph has $y = 7$ as a horizontal asymptote on the right side only.

When you match the functions with these graphs, add (if any) horizontal asymptotes to the pictures.

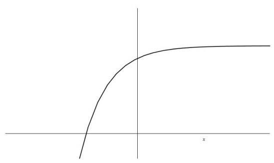


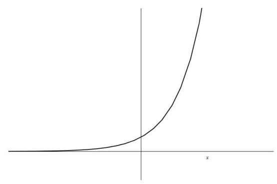


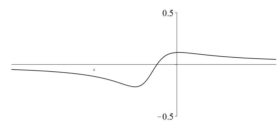




$$y = \frac{x^3+4x+9}{x^2+4} \qquad y = \frac{\sqrt{8x^2+4}}{x+2} \qquad y = -7x^4 + x^3 - 12x + 20 \qquad y = \frac{(x+1)^4}{x^4+3x^2+7x+10}$$







$$y = 7 - e^{-x} \qquad y = 3e^x \qquad y = \frac{x+3}{x^2+8x+26}$$