

Definition 17.1.1 (The Sinusoidal Function). Let A , B , C and D be fixed constants, where A and B are both positive. Then we can form the new function

$$y = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

which is called a sinusoidal function. The four constants can be interpreted graphically as indicated:

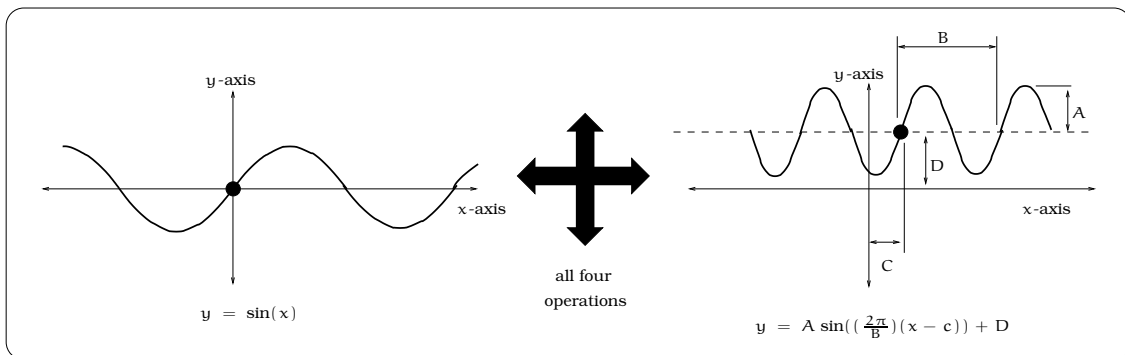


Figure 17.7: Putting it all together for the sinusoidal function.

17.1.1 How to roughly sketch a sinusoidal graph

Important Procedure 17.1.2. Given a sinusoidal function in the standard form

$$y = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

once the constants A , B , C , and D are specified, any graphing device can produce an accurate graph. However, it is pretty straightforward to sketch a rough graph by hand and the process will help reinforce the graphical meaning of the constants A , B , C , and D . Here is a “five step procedure” one can follow, assuming we are given A , B , C , and D . It is a good idea to follow Example 17.1.3 as you read this procedure; that way it will seem a lot less abstract.

1. Draw the horizontal line given by the equation $y = D$; this line will “split” the graph of $y = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D$ into symmetrical upper and lower halves.
2. Draw the two horizontal lines given by the equations $y = D \pm A$. These two lines determine a horizontal strip inside which the graph of the

sinusoidal function will oscillate. Notice, the points where the sinusoidal function has a maximum value lie on the line $y = D + A$. Likewise, the points where the sinusoidal function has a minimum value lie on the line $y = D - A$. Of course, we do not yet have a prescription that tells us where these maxima (peaks) and minima (valleys) are located; that will come out of the next steps.

3. Since we are given the period B , we know these important facts: (1) The period B is the horizontal distance between two successive maxima (peaks) in the graph. Likewise, the period B is the horizontal distance between two successive minima (valleys) in the graph. (2) The horizontal distance between a maxima (peak) and the successive minima (valley) is $\frac{1}{2}B$.
4. Plot the point (C, D) . This will be a place where the graph of the sinusoidal function will cross the mean line $y = D$ on its way up from a minima to a maxima. This is not the only place where the graph crosses the mean line; it will also cross at the points obtained from (C, D) by horizontally shifting by any integer multiple of $\frac{1}{2}B$. For example, here are three places the graph crosses the mean line: (C, D) , $(C + \frac{1}{2}B, D)$, $(C + B, D)$
5. Finally, midway between (C, D) and $(C + \frac{1}{2}B, D)$ there will be a maxima (peak); i.e. at the point $(C + \frac{1}{4}B, D + A)$. Likewise, midway between $(C + \frac{1}{2}B, D)$ and $(C + B, D)$ there will be a minima (valley); i.e. at the point $(C + \frac{3}{4}B, D - A)$. It is now possible to roughly sketch the graph on the domain $C \leq x \leq C + B$ by connecting the points described. Once this portion of the graph is known, the fact that the function is periodic tells us to simply repeat the picture in the intervals $C + B \leq x \leq C + 2B$, $C - B \leq x \leq C$, etc.

To make sense of this procedure, let's do an explicit example to see how these five steps produce a rough sketch.

Example 17.1.3. The temperature (in $^{\circ}\text{C}$) of Adri-N's dorm room varies during the day according to the sinusoidal function $d(t) = 6 \sin\left(\frac{\pi}{12}(t - 11)\right) + 19$, where t represents hours after midnight. Roughly sketch the graph of $d(t)$ over a 24 hour period.. What is the temperature of the room at 2:00 pm? What is the maximum and minimum temperature of the room?

Solution. We begin with the rough sketch. Start by taking an inventory of the constants in this sinusoidal function:

$$d(t) = 6 \sin\left(\frac{\pi}{12}(t - 11)\right) + 19 = A \sin\left(\frac{2\pi}{B}(t - C)\right) + D.$$

Conclude that $A = 6$, $B = 24$, $C = 11$, $D = 19$. Following the first four steps of the procedure outlined, we can sketch the lines $y = D = 19$, $y = D \pm A = 19 \pm 6$ and three points where the graph crosses the mean line (see Figure 17.8).

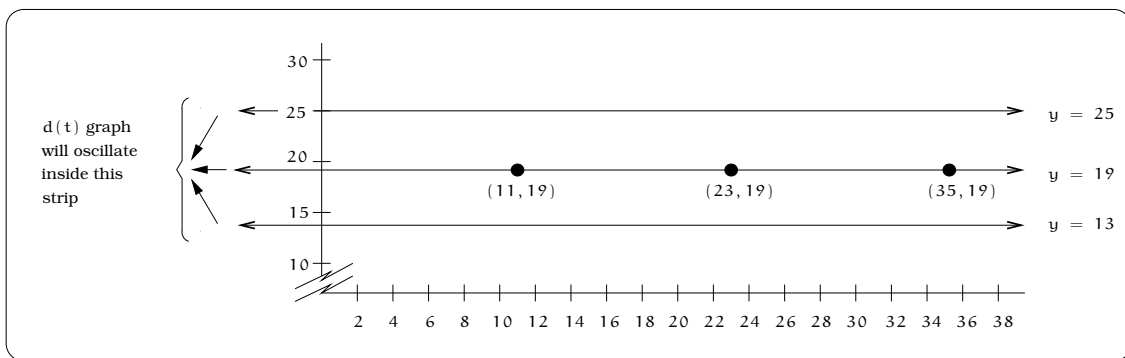


Figure 17.8: Sketching the mean D and amplitude A .

According to the fifth step in the sketching procedure, we can plot the maxima $(C + \frac{1}{4}B, D + A) = (17, 25)$ and the minima $(C + \frac{3}{4}B, D - A) = (29, 13)$. We then “connect the dots” to get a rough sketch on the domain $11 \leq t \leq 35$.

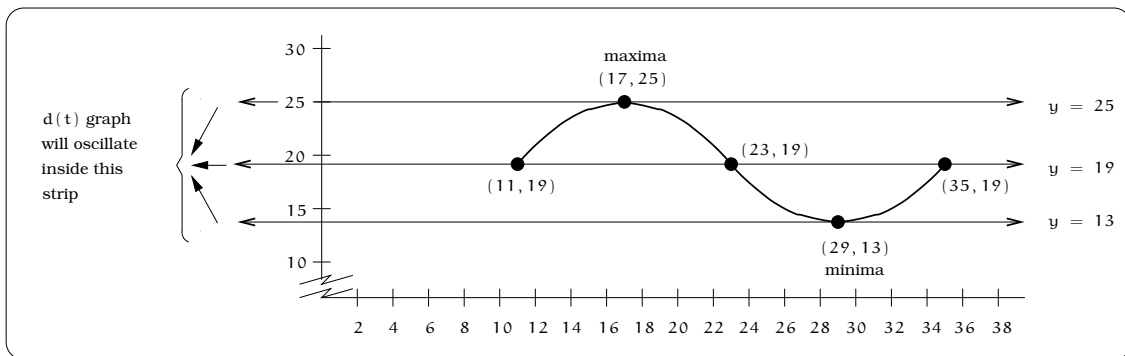


Figure 17.9: Visualizing the maximum and minimum over one period.

Finally, we can use the fact the function has period 24 to sketch the graph to the right and left by simply repeating the picture every 24 horizontal units.

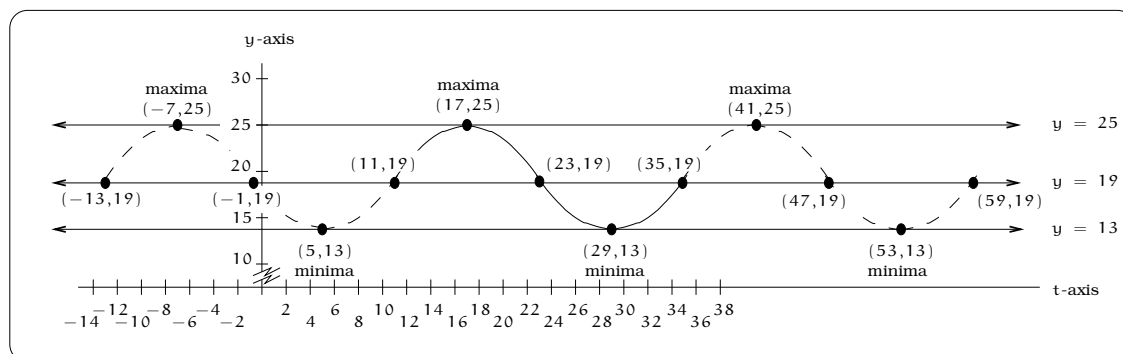


Figure 17.10: Repeat sketch for every full period.

We restrict the picture to the domain $0 \leq t \leq 24$ and obtain the computer generated graph pictured in Figure 17.11; as you can see, our rough graph is very accurate. The temperature at 2:00 p.m. is just $d(14) = 23.24^\circ\text{C}$. From the graph, the maximum value of the function will be $D + A = 25^\circ\text{C}$ and the minimum value will be $D - A = 13^\circ\text{C}$.

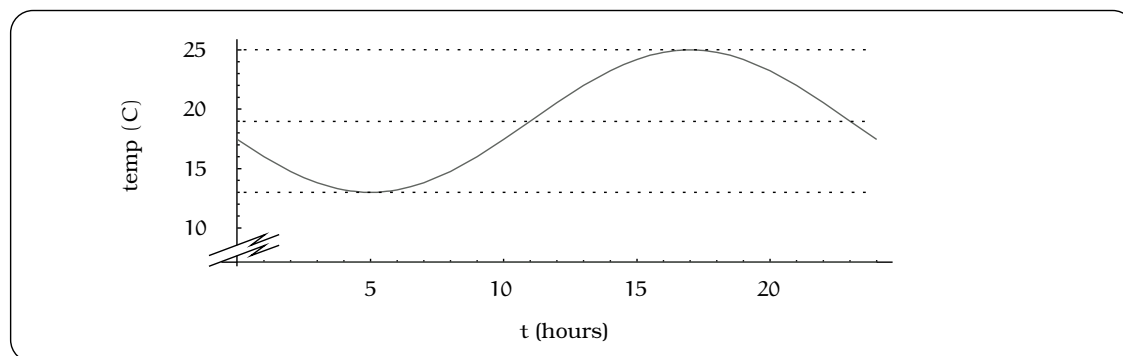


Figure 17.11: The computer generated solution.

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17.1.2 Functions not in standard sinusoidal form

Any time we are given a trigonometric function written in the standard form

$$y = A \sin \left(\frac{2\pi}{B}(x - C) \right) + D,$$

for constants A , B , C , and D (with A and B positive), the summary in Definition 17.1.1 tells us everything we could possibly want to know

about the graph. But, there are two ways in which we might encounter a trigonometric type function that is not in this standard form:

- The constants A or B might be negative. For example, $y = -2 \sin(2x - 7) - 3$ and $y = 3 \sin(-\frac{1}{2}x + 1) + 4$ are examples that fail to be in standard form.
- We might use the cosine function in place of the sine function. For example, something like $y = 2 \cos(3x + 1) - 2$ fails to be in standard sinusoidal form.

Now what do we do? Does this mean we need to repeat the analysis that led to Definition 17.1.1? It turns out that if we use our trig identities just right, then we can move any such equation into standard form and read off the amplitude, period, shift and mean. In other words, equations that fail to be in standard sinusoidal form for either of these two reasons will still define sinusoidal functions. We illustrate how this is done by way of some examples:

Examples 17.1.4.

- (i) Start with $y = -2 \sin(2x - 7) - 3$, then here are the steps with reference to the required identities to put the equation in standard form:

$$\begin{aligned} y &= -2 \sin(2x - 7) - 3 \\ &= 2(-\sin(2x - 7)) - 3 \\ &= 2 \sin(2x - 7 + \pi) + (-3) && \text{Fact 16.2.5 on page 219} \\ &= 2 \sin\left(\frac{2\pi}{\pi}\left(x - \left[\frac{7 - \pi}{2}\right]\right)\right) + (-3). \end{aligned}$$

This function is now in the standard form of Definition 17.1.1, so it is a sinusoidal function with shift $C = \frac{7 - \pi}{2} = 1.93$, mean $D = -3$, amplitude $A = 2$ and period $B = \pi$.

- (ii) Start with $y = 3 \sin(-\frac{1}{2}x + 1) + 4$, then here are the steps with reference to the required identities to put the equation in standard form:

$$\begin{aligned} y &= 3 \sin\left(-\frac{1}{2}x + 1\right) + 4 \\ &= 3 \sin\left(-\left(\frac{1}{2}x - 1\right)\right) + 4 \\ &= 3\left(-\sin\left(\frac{1}{2}x - 1\right)\right) + 4 && \text{Fact 16.2.4 on page 219} \\ &= 3 \sin\left(\frac{1}{2}x - 1 + \pi\right) + 4 && \text{Fact 16.2.5 on page 219} \\ &= 3 \sin\left(\frac{2\pi}{4\pi}\left(x - [2 - 2\pi]\right)\right) + 4 \end{aligned}$$

This function is now in the standard form of Definition 17.1.1, so it is a sinusoidal function with shift $C = 2 - 2\pi$, mean $D = 4$, amplitude $A = 3$ and period $B = 4\pi$.

- (iii) Start with $y = 2 \cos(3x + 1) - 2$, then here are the steps to put the equation in standard form. A key simplifying step is to use the identity: $\cos(t) = \sin(\frac{\pi}{2} + t)$.

$$\begin{aligned} y &= 2 \cos(3x + 1) - 2 \\ &= 2 \sin\left(\frac{\pi}{2} + 3x + 1\right) - 2 \\ &= 2 \sin\left(3x - \left[-1 - \frac{\pi}{2}\right]\right) + (-2) \\ &= 2 \sin\left(\frac{2\pi}{\left(\frac{2\pi}{3}\right)}\left(x - \frac{1}{3}\left[-1 - \frac{\pi}{2}\right]\right)\right) + (-2) \end{aligned}$$

This function is now in the standard form of Definition 17.1.1, so it is a sinusoidal function with shift $C = \frac{1}{3}\left[-1 - \frac{\pi}{2}\right]$, mean $D = -2$, amplitude $A = 2$ and period $B = \frac{2\pi}{3}$.

17.2 Examples of sinusoidal behavior

Problems involving sinusoidal behavior come in two basic flavors. On the one hand, we could be handed an explicit sinusoidal function

$$y = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

and asked various questions. The answers typically require either direct calculation or interpretation of the constants. Example 17.1.3 is typical of this kind of problem. On the other hand, we might be told a particular situation is described by a sinusoidal function and provided some data or a graph. In order to further analyze the problem, we need a “formula”, which means finding the constants A , B , C , and D . This is a typical scenario in a “mathematical modeling problem”: the process of observing data, THEN obtaining a mathematical formula. To find A , take half the difference between the largest and smallest values of $f(x)$. The period B is most easily found by measuring the distance between two successive maxima (peaks) or minima (valleys) in the graph. The mean D is the average of the largest and smallest values of $f(x)$. The shift C (which is usually the most tricky quantity to get your hands on) is found by locating a “reference point”. This “reference point” is a location where the graph crosses the mean line $y = D$ on its way up from a minimum to a maximum. The funny thing is that the shift C is NOT unique; there are an infinite number of correct choices. One choice that will work is

$C = (\text{x-coordinate of a maximum}) - \frac{B}{4}$. Any other choice of C will differ from this one by a multiple of the period B .

$$A = \frac{\text{max value} - \text{min value}}{2}$$

$$B = \text{distance between two successive peaks (or valleys)}$$

$$C = \text{x-coordinate of a maximum} - \frac{B}{4}$$

$$D = \frac{\text{max value} + \text{min value}}{2}.$$

Example 17.2.1. Assume that the number of hours of daylight in Seattle is given by a sinusoidal function $d(t)$ of time. During 1994, assume the longest day of the year is June 21 with 15.7 hours of daylight and the shortest day is December 21 with 8.3 hours of daylight. Find a formula $d(t)$ for the number of hours of daylight on the t^{th} day of the year.

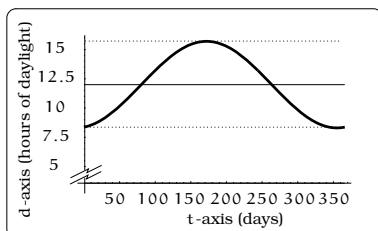


Figure 17.12: Hours of daylight in Seattle in 1994.

Solution. Because the function $d(t)$ is assumed to be sinusoidal, it has the form $y = A \sin\left(\frac{2\pi}{B}(t - C)\right) + D$, for constants A , B , C , and D . We simply need to use the given information to find these constants. The largest value of the function is 15.7 and the smallest value is 8.3. Knowing this, from the above discussion we can read off:

$$D = \frac{15.7 + 8.3}{2} = 12 \qquad A = \frac{15.7 - 8.3}{2} = 3.7.$$

To find the period, we need to compute the time between two successive maximum values of $d(t)$. To find this, we can simply double the time length of one-half period, which would be the length of time between successive maximum and minimum values of $d(t)$. This gives us the equation

$$B = 2(\text{days between June 21 and December 21}) = 2(183) = 366.$$

Locating the final constant C requires the most thought. Recall, the longest day of the year is June 21, which is day 172 of the year, so

$$C = (\text{day with max daylight}) - \frac{B}{4} = 172 - \frac{366}{4} = 80.5.$$

In summary, this shows that

$$d(t) = 3.7 \sin\left(\frac{2\pi}{366}(t - 80.5)\right) + 12.$$

A rough sketch, following the procedure outlined above, gives this graph on the domain $0 \leq t \leq 366$; we have included the mean line $y = 12$ for reference. \square