Explorations of Rigid Motions and Congruence

James King
University of Washington
Department of Mathematics
king@uw.edu
http://www.math.washington.edu/~king

The "Plan"

- In this session, we will explore exploring.
- We have a big math toolkit of transformations to consider.
- We have some physical objects that can serve as a hands-on manipulative toolkit.
- We have geometry concepts and relationships to think about.
- And we have the point of view of geometry in the Common Core State Standards to reflect on.
- And we want to think about sense-making and reasoning throughout.

Introductory Activity

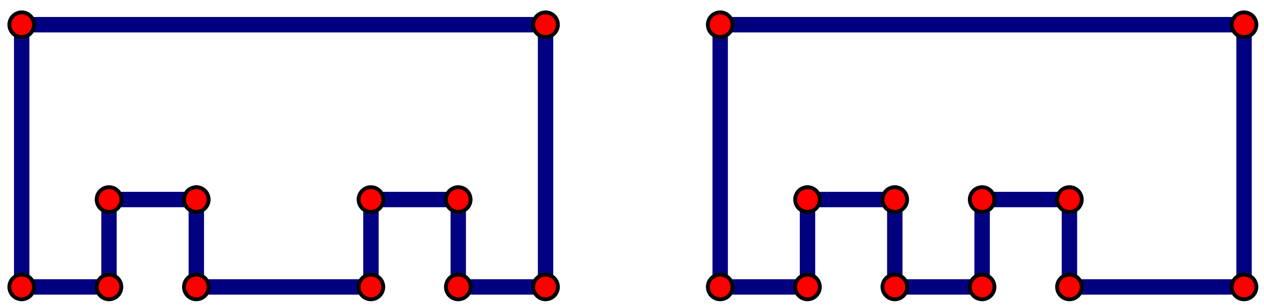
- Think quietly for a minute about how you define for your students what it means for two figures in the plane to be congruent.
- Then privately write down this definition on a piece of paper.
- When everyone is finished, we will discuss.

Points to Ponder

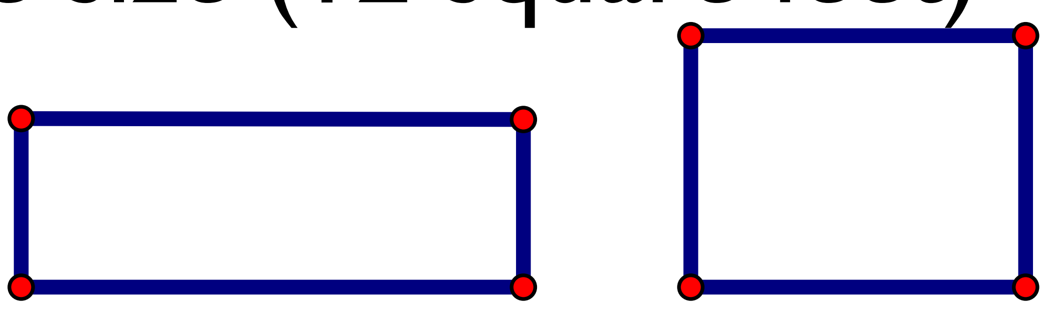
- Does your definition use mathematically undefined terms from ordinary language, such as "same size, same shape" or "pick up and move" or "superimpose"?
- Does your definition apply to *any* figure in the plane, or just to polygons?
- Does your concept of congruence include some additional hidden assumptions or rules not spelled out in the definition?

Congruent?

- Equal side lengths, equal angles

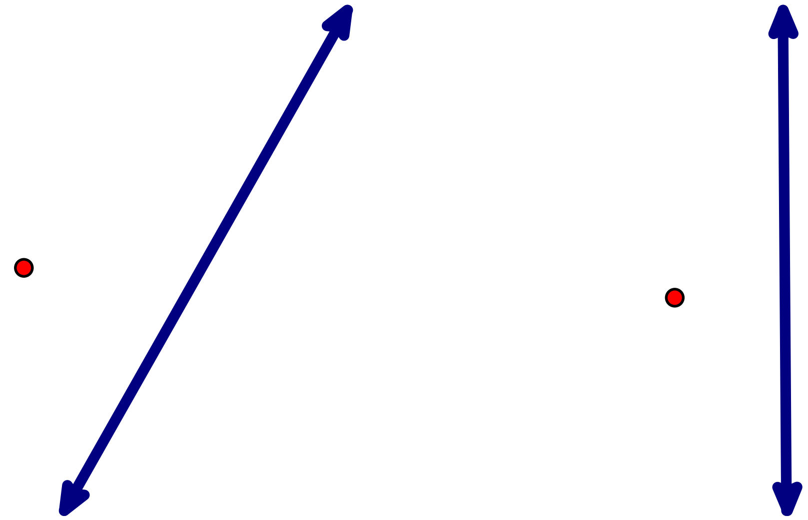


- Two garden plots, same shape (rectangular) and same size (12 square feet)



Congruent?

- Two circles? What angles are supposed to be equal?
- Two parabolas? The length is infinite.
- Two disconnected figures, each consisting of a line and a point not on the line.



Common Core Approach

- Grade 8: *Verify experimentally the properties of rotations, reflections, and translations*
- Grade 8: *Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.*
- High School: *Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.*

Rigid Motions?

- The language of CCSS for middle school refers to sequences of rotations, reflections, and translations, while in high school it speaks of rigid motions.
- In CCSS, a rigid motion is defined to be such a sequence. It is assumed, essentially as an axiom, that a rigid motion preserves distance and angle measure.
- We will talk more later how this fits in with other foundational approaches to geometry.

Solid Definition of Congruence

- The rigid motion definition is a clear, unambiguous concept. This gives meaning to congruence of any shapes, from polygons to ellipses and parabolas, to fractals with an easy extension to digital photos.

- This contrasts with the "definition" of congruence in many secondary texts: lots of intuition about cutting out and moving and "same size same shape" but no well-defined general concept, just tests for triangles and then ad hoc definitions for other shapes.

- Note that a rigid motion is not the same as superimposition of figures (cut out and move); rigid motions are defined for the whole plane, not just for points in the figure. The whole plane moves and nothing is cut out. This is sound mathematics that lays groundwork for more advanced math.

Our Transformational Case of Characters

- Line Reflection
- Point Reflection (a rotation)
- Translation
- Rotation
- Compositions of any of the above

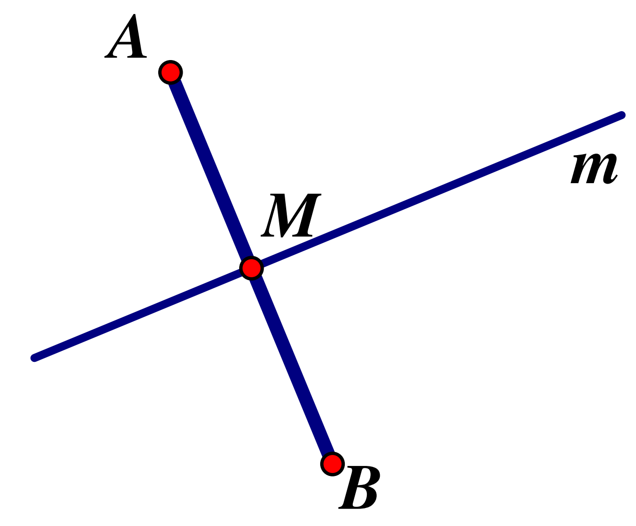
Our Physical Toolkit

- Patty paper
- Semi-reflective plastic mirrors
- Graph paper
- Ruled paper
- Card Stock
- Dot paper
- Scissors, rulers, protractors

Line Reflection

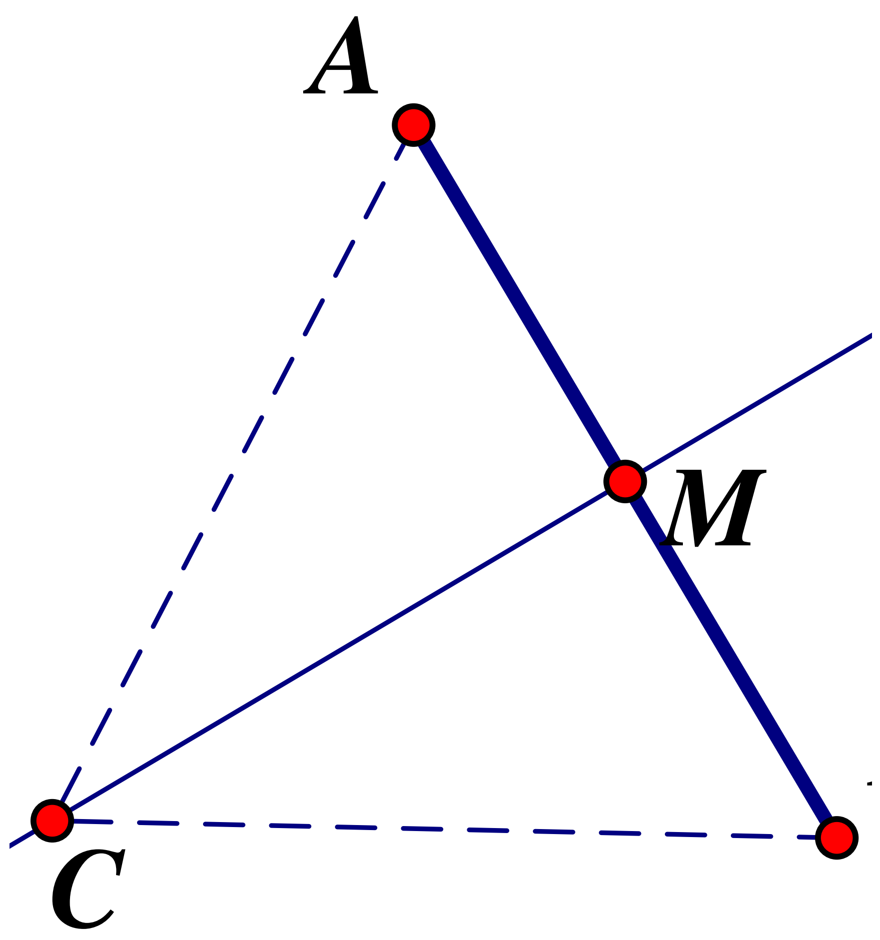
- For any line m in the plane, there is defined a transformation of the plane called "line reflection across m".

- If A is a point in the plane, the reflection of A across m is the point B such that (1) segment AB is perpendicular to m and (2) the intersection point M of AB and m is the midpoint of AB.



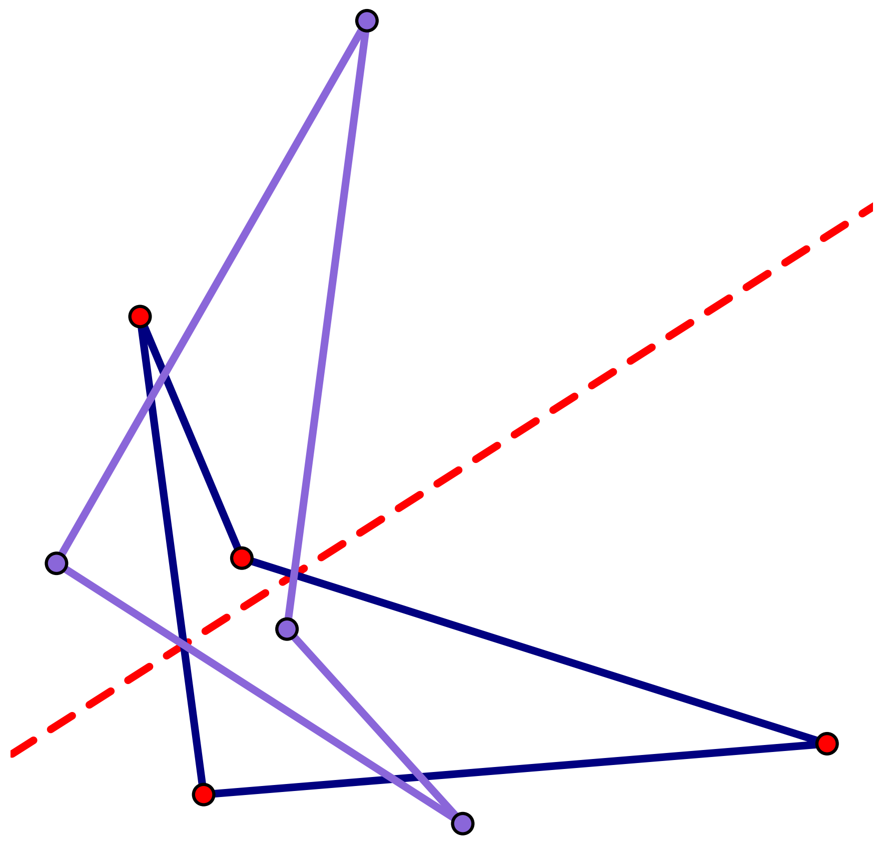
Line-Reflecting a Figure

- As a first task, we will try out tools for line reflection of a point A to a point B. Then reflecting a shape.
- Suggest that you try the **semi-reflective mirrors** and the **patty paper** for folding and tracing. Also, graph paper is an option. Also, regular paper and cut-outs
- Note that pencils and overhead pens work on patty paper but not ballpoints. Also note that overhead dots are easier to see with the mirrors.
- Can we (or your students) conclude from your tool that the mirror line is the perpendicular bisector of AB?



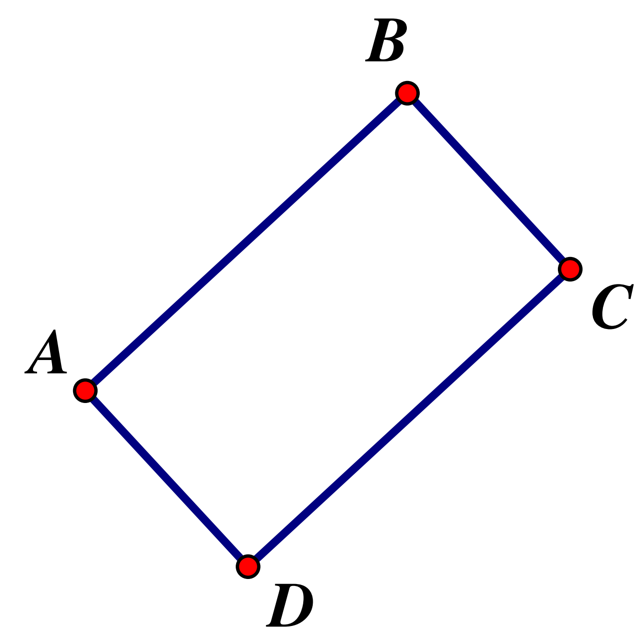
Which tools best let you draw this reflection?

- When reflecting shapes, consider how to reflect some polygon when it is not all on one side of the mirror line.
- Otherwise students may be the wrong idea that reflection only works if the whole figure is on one side of the mirror line.



Line Symmetry

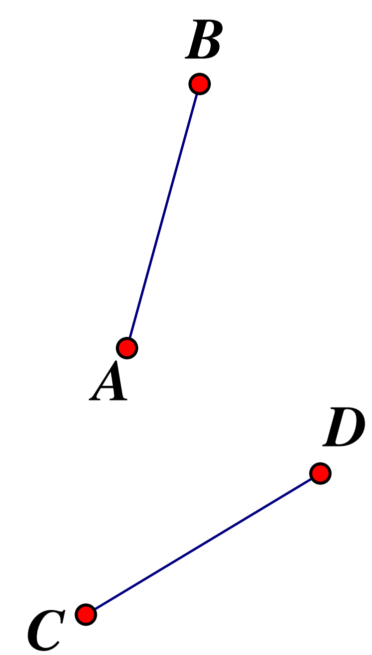
- For a figure S in the plane, a line m is a line of symmetry of S if the reflection of S in m is exactly S itself.
- For a rectangle ABCD, what are the lines of symmetry?
- Reflect ABCD across the diagonal line AC. Is this a line of symmetry?
- What can you say about a triangle if it has a line of symmetry? What is this line?
- If a quadrilateral ABCD has AC as a line of symmetry, what kind of quadrilateral is it?



Congruent Line Segments

- Here's food for thought: In CCSS, given the definition of congruence, we cannot assume without proof that two line segments of the same length are congruent. We must show that for any two such segments we can move one to the other by a sequence of rigid motions!

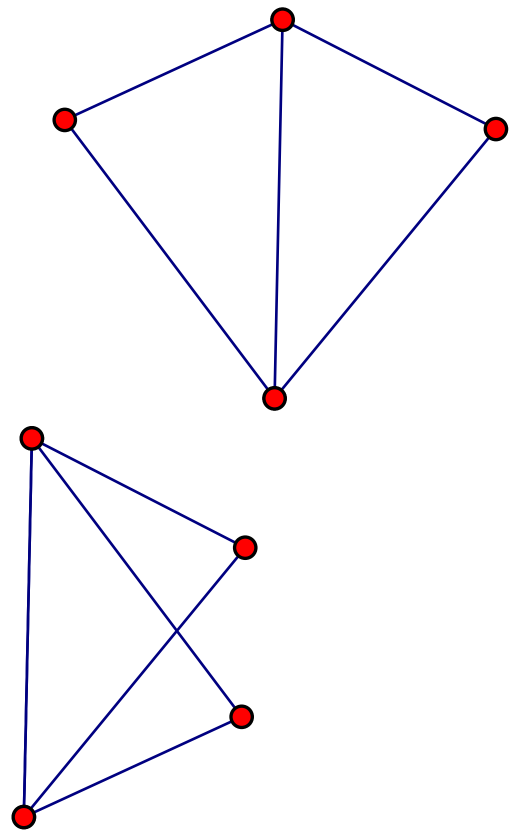
- So, as an exercise, draw two line segments of the same length and perform a sequence of reflections that will take one to the other. Do you think this can always be done? How many reflections does it take?



Proving SAS etc.

- To prove SAS, you can build on what we have done to move one side of the first triangle onto the second. Then either you are done already, or you get a figure like one of these.

- Can you justify the final step?

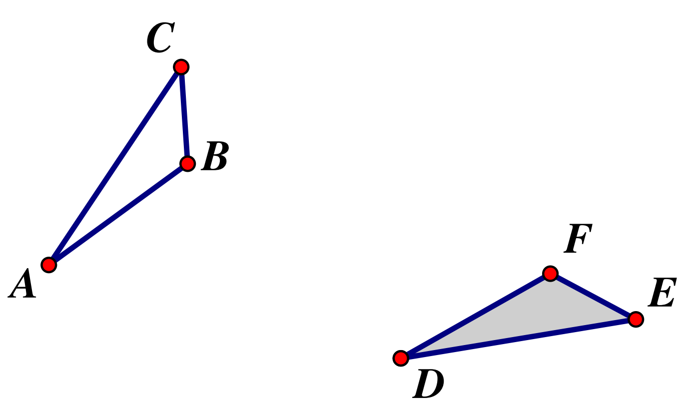
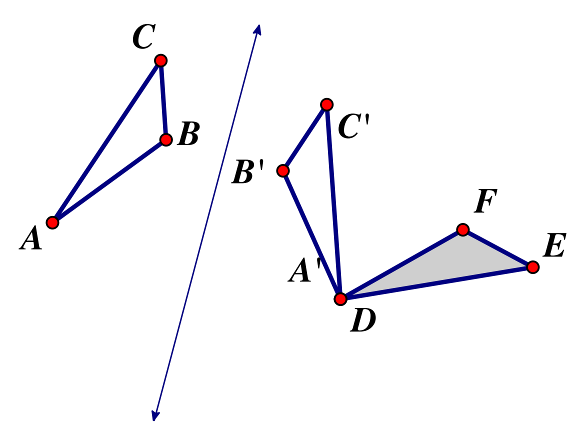
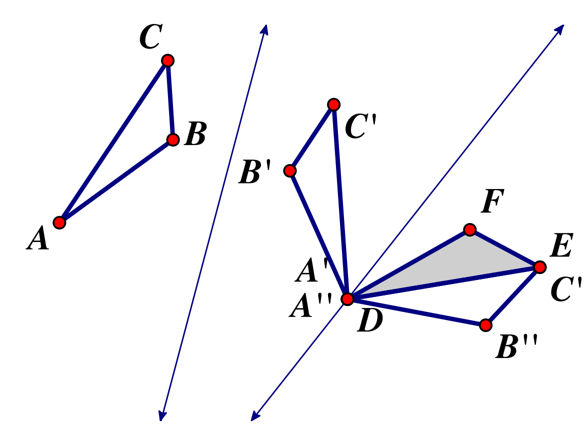


SAS from Rigid Motions

- SAS: Given two triangles ABC and DFE so that angle BAC and angle FDE have equal measure, length AB = length DF, and length AC = length DE, then triangle ABC is congruent to triangle DFE.
- How do we prove this with rigid motions? Find a sequence of rigid motions that will take one triangle to the other given these assumptions.
- There is a choice of ways to do this. Start with a translation that takes A to D ... or start with a line reflection that takes A to D, or one could move A to D by a rotation. Since we are working with line reflections, we start with a line reflection.

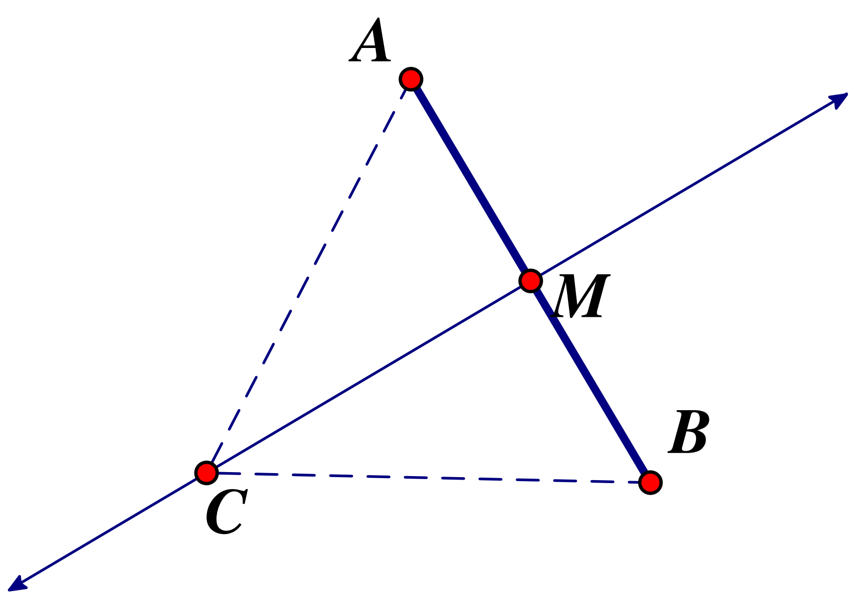
Executive Summary of the Proof of SAS

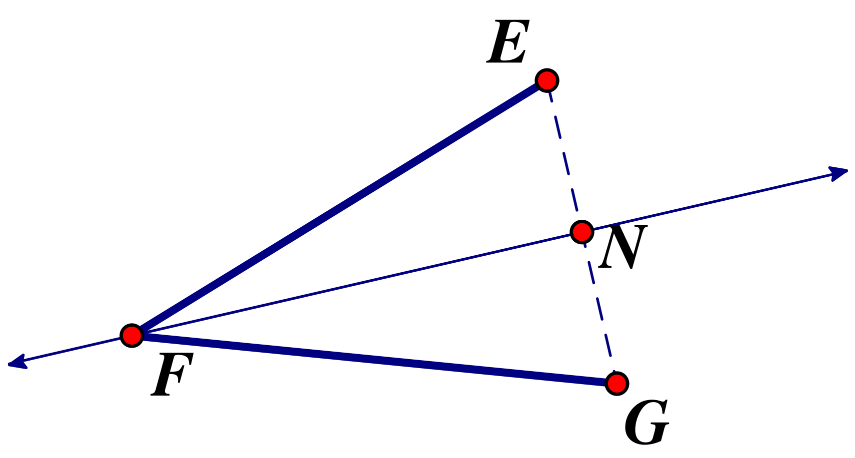
- Assume angle CAB = angle EDF; AB = DF; AC = DE.
- Here are the steps in a proof, but they are not a proof, since we need reasons why the steps work.
- The reasons will be explored on the next slide.

Step 1: Reflect A to D. ABC is reflected to A’B’C’, with A’ = D.	Step 2: Reflect C’ to E in a line through D. A’B’C’ is reflected to A’’B’’C’’, with A’’ = D and C’’ = E. If B’’ = F, stop.	Step 3: Reflect B’’ to F in line DE. A’’B’’C’’ is reflected to A’’’B’’’C’’’, with A’’’ = D, C’’’ = E, and B’’’ = E
		

Two Basic Theorems

- **Proposition 1**: If a point A is line reflected to point B, the line of reflection is the **perpendicular bisector** of segment AB.
- This proposition follows immediately from our definition of line reflection. With other definitions (discussed later), this is a theorem to prove.
- **Proposition 2**: If a segment FE is congruent to FG, then the **angle bisector of angle EFG** reflects point E to point G.
- **Proof.** Since line reflection preserves angle measure, the reflection in the bisector of the ray FE is ray FG. Let E' be the reflection of point E. Since the segments FE' and FG are congruent and lie on the same ray, the point E' and point G are the same.
- **Corollary.** In this figure, since E is reflected to G, the triangle EFG is **isosceles** and the angle bisector of angle EFG is the perpendicular bisector of EG.

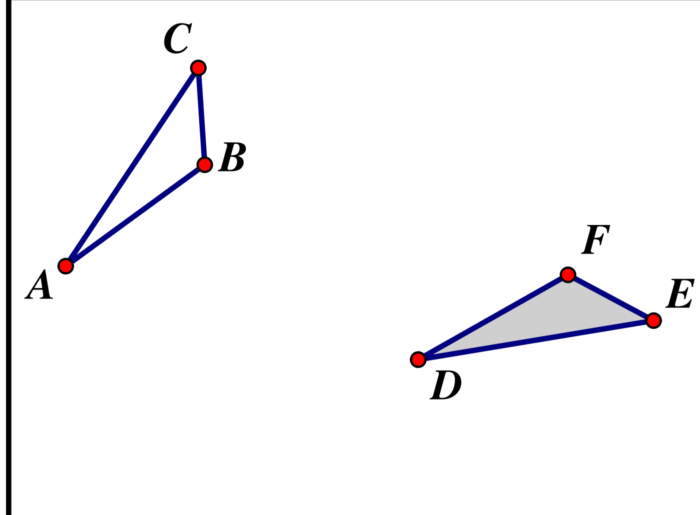




Proof of Step 1

- Let m be the perpendicular bisector of segment AD, then reflection in m maps A to D.

Step 1: Reflect A to D. ABC is reflected to A’B’C’, with A’ = D.

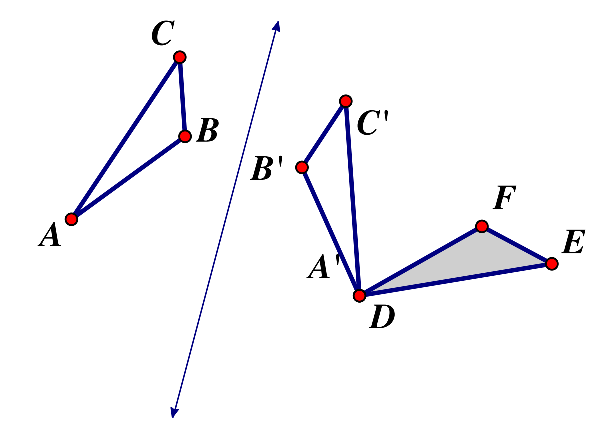


Proof of Step 2

- There are two cases. One possibility is that point C' is the same as point E. In this case, we skip this step and go directly to Step 3.

- In the more likely case, point C' and E are different points. Since the segments DC' and DE are equal, then by Proposition 2, the angle bisector of angle C'DE maps C' to E. And since the line passes through D, the point D is fixed.

Step 2: Reflect C' to E in a line through D. A'B'C' is reflected to A''B''C'', with A'' = D and C'' = E. If B'' = F, stop.

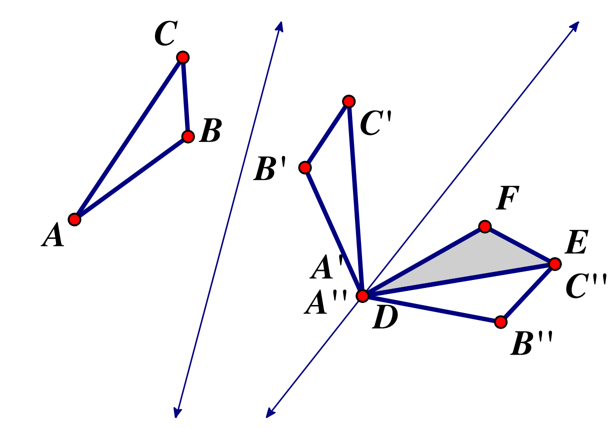


Proof of Step 3

- At this point, we have two triangles, B”DE and FDE with a common side. We also know that DB” is congruent to DF and that angles B”DE and FDE are congruent.

- There are two cases, either F and B” are the same point, or not. In the first case we have finished the proof after two steps. In the second, we notice that ray DE is the angle bisector of angle B”DF, so that by Theorem 2, the point B” is reflected across DE to F. And the points D and E are fixed. Thus this third reflection maps the triangle B”DE to triangle FDE, so the SAS theorem is proved after 3 steps.

Step 3: Reflect B’’ to F in line DE. A’’B’’C’’ is reflected to A’’’B’’’C’’’, with A’’’ = D, C’’’ = E, and B’’’ = E



What have we proved?

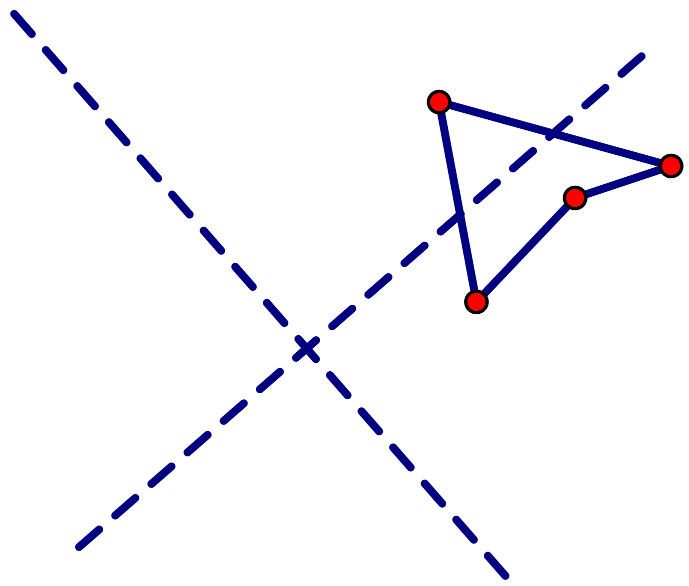
- First of all, we just proved our old friend the SAS congruence criterion for triangles. In some approaches to geometry, this is an axiom, in others it is a theorem. With a little more work, we can also prove ASA and SSS.
- One important point is that now that we have this tool, we can use it. We do not have to explicitly use rigid motions in every proof just because we are following CCSS. Having said this, we should also be aware of situations where having rigid motions as a tool can be very powerful and sense-making. It is a good time to re-think our ideas. But we do not have to discard everything we are used to doing in geometry.
- One other thing: We have proved that any two distinct congruent triangles can be related by a sequence of 1, 2, or 3 line reflections. With a bit more work, we can see that this is true for any two congruent figures, no matter how complicated.

Exploration: Reflections in Perpendicular Lines

- We will now see what happens if we reflect a figure twice, first in any line and then in a line perpendicular to the first.

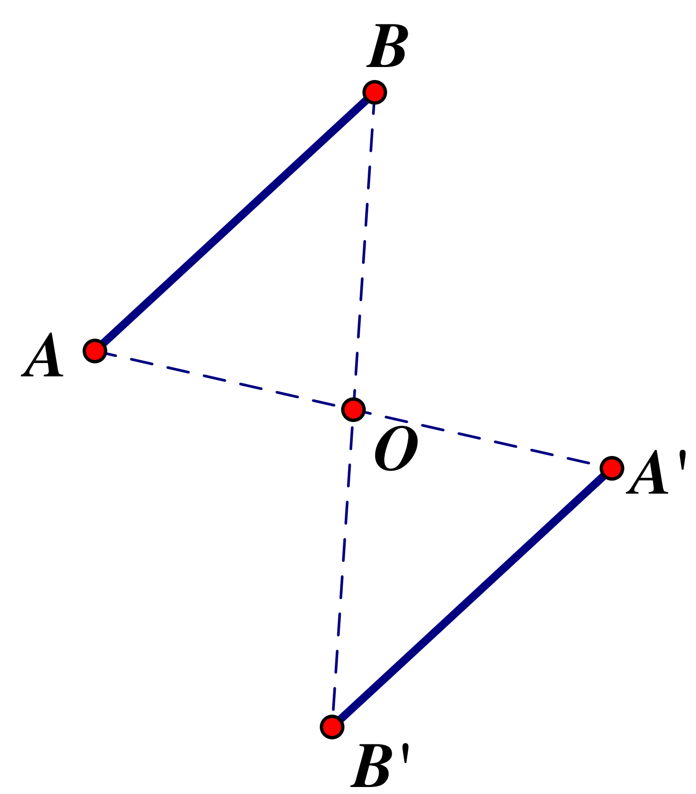
- A good tool for this would be patty paper folding, though mirrors also work. Fold a paper twice so that the folds are perpendicular. Draw any figure, and use tracing to reflect the figure first in one line and then reflect the image figure in the second.

- How is the original figure related to the second reflection? What happens if you connect corresponding points with lines? What point(s) n the plane do not move under this sequence of reflections?



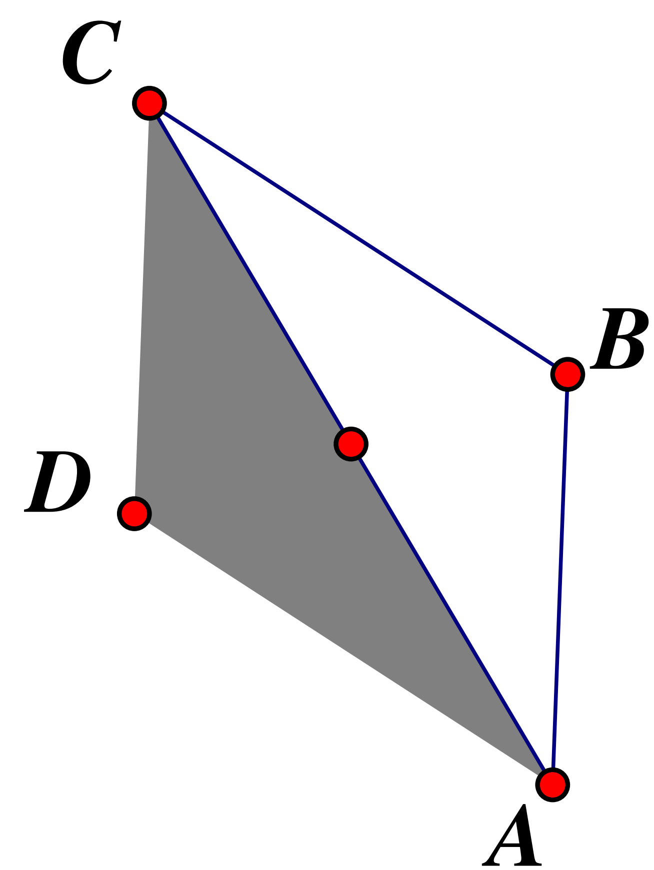
Point Reflection (half-turn)

- A *point reflection*, or a *half-turn* is an important special case of rotation: rotation by 180 degrees.
- It is not difficult to apply a half-turn to a point A with a straightedge and a piece of card (or a ruler) –or with patty paper or other tools.
- Point-reflect two points A and B with center O. **Can you justify the claim that line A'B' is parallel to line AB?** Does this mean also that line AB' is parallel to line A'B?
- Houston, we have a parallelogram!



Parallelogram from 3 points

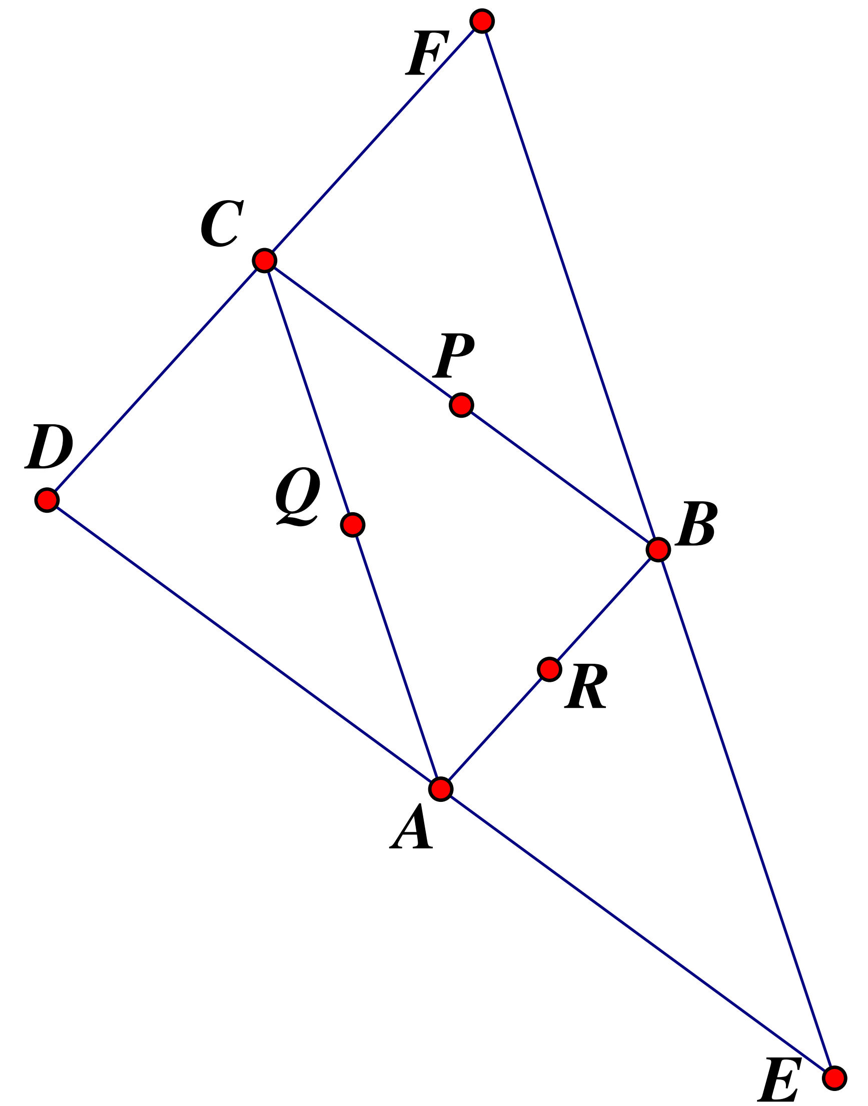
- Given a triangle ABC, find the center of a half-turn that will construct a point D so that ABCD is a parallelogram.
- Or else that ABDC is a parallelogram, or ...
- How does this figure compare with what you would get from a line reflection?
- A figure has a ***point symmetry*** if there is a half-turn that maps the figure onto itself. Does every parallelogram have a point symmetry? What is the center of the symmetry?



Hmm!

- If you create 3 parallelograms from ABC all in the same figure, this is what you get! A rich figure.

- Can you find any examples here of triangle pairs that are related by a composition of two half-turns?



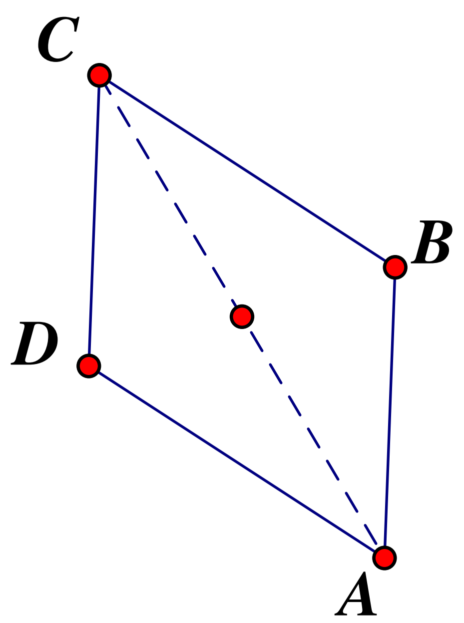
Proving Parallelogram Properties with Half-turns

- There are several standard theorems about parallelograms that can be proved with half-turns. These proofs seems clearer and more visual than the usual ones.

- Let ABCD be a parallelogram. Let M be the midpoint of AC. Then apply a half-turn centered at M to this figure. **Why are these statements true?**

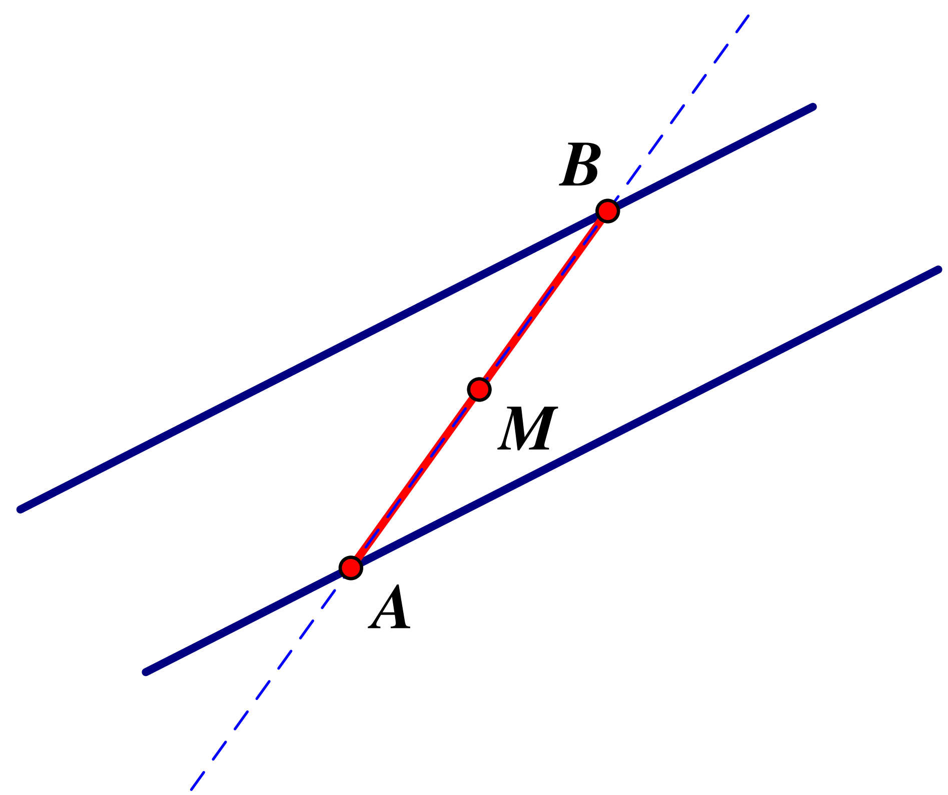
- C' is the image of A. The line CD is the image of line AB. The line CB is the image of line AD. M is the midpoint of CD. Angle A is congruent to angle C. Angle B is congruent to angle D. Side AB is congruent to side CD. Side BC is congruent to side DA.

- Also, one can start with a quadrilateral ABCD and assume opposite sides are parallel and equal and prove that ABCD is a parallelogram.



Transversals and Point Reflection

- Given two parallel lines and a transversal line, there are a number of familiar theorems about which angles are equal.
- One can use point reflection to prove these theorems. What is the center of a point symmetry of this figure?
- Equally important, this point of view makes it very comprehensible why certain angle pairs (e.g., the alternate interior angles) are congruent.

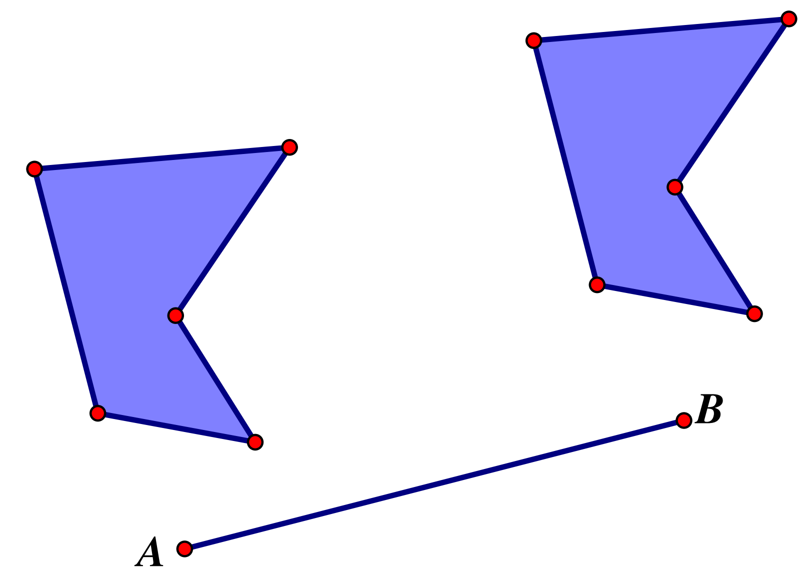


Exploration: Double Reflection in Parallel lines

- Draw two parallel lines *m* and *n* on a piece of patty paper, not too far apart.
- Pick any point A. Reflect A in m to get A'. Then reflect A' in n to get A''.
- This double reflection is a rigid motion. Draw a simple figure (triangle or quadrilateral ABCD , perhaps) and double reflect the figure. See what you can observe and conjecture.
- Make a list of the (conjectured) properties of this transformation that you observe.

A Double Reflection in Parallel Lines is aTranslation

- Translation is often viewed as the simplest of our rigid motions. To translate a figure, just slide it without rotating.
- But it is not so easy to figure out a simple way to say this mathematically, without using geometry theorems that will come later.
- I am not sure what would be the best definition to use. What do you think? (Some ideas on the next slide.)

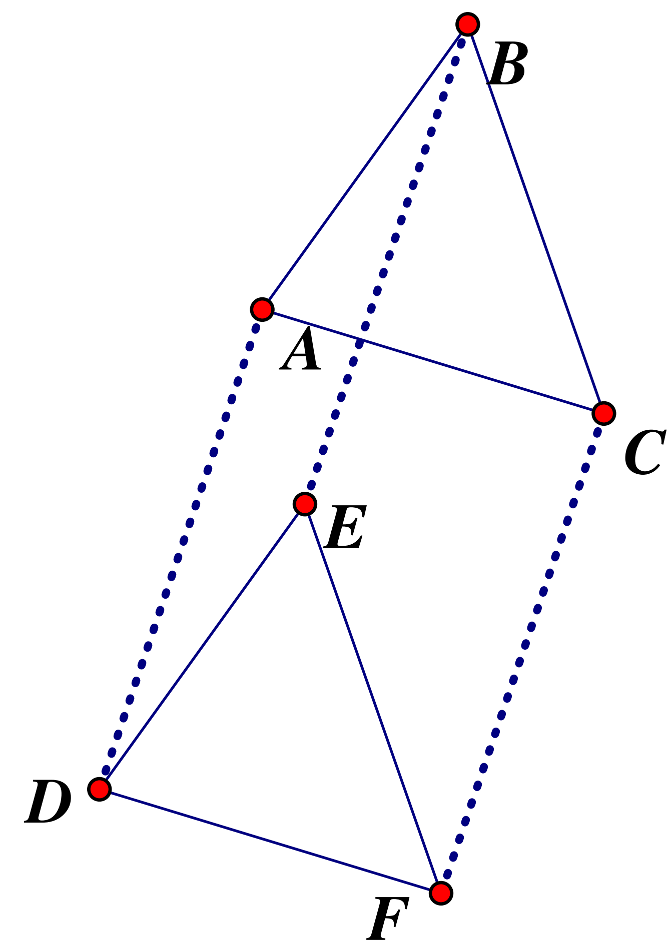


Possible Definitions of a Translation

- A translation is a double reflection in parallel lines.
- A translation is a double point reflection.
- A translation is a rigid motion of the plane so that for every point A and its image A’ the distance from A to A’ is the same.
- A translation is a rigid motion of the plane with no fixed point, so that every line is mapped to itself or a parellel line.
- Given points A and B in the plane, the translation with defined by vector AB is the rigid motion that takes a point C to a point D such that ABDC is a parallelogram (or a collapsed parallelogram if C is on line AB).

Translation: Hands-On is Harder than it looks

- Translations are commonly viewed as the easy transformations to model hands-on: just slide!
- But how can you be sure that your freehand slide does not have some rotation in it? We need a careful slide.
- Suggestion. We know how to use half-turns to construct parallelograms. Can you use this a a practical way to translate one polygon to another?
- Other ideas? Tracing with patty paper? Graph paper?



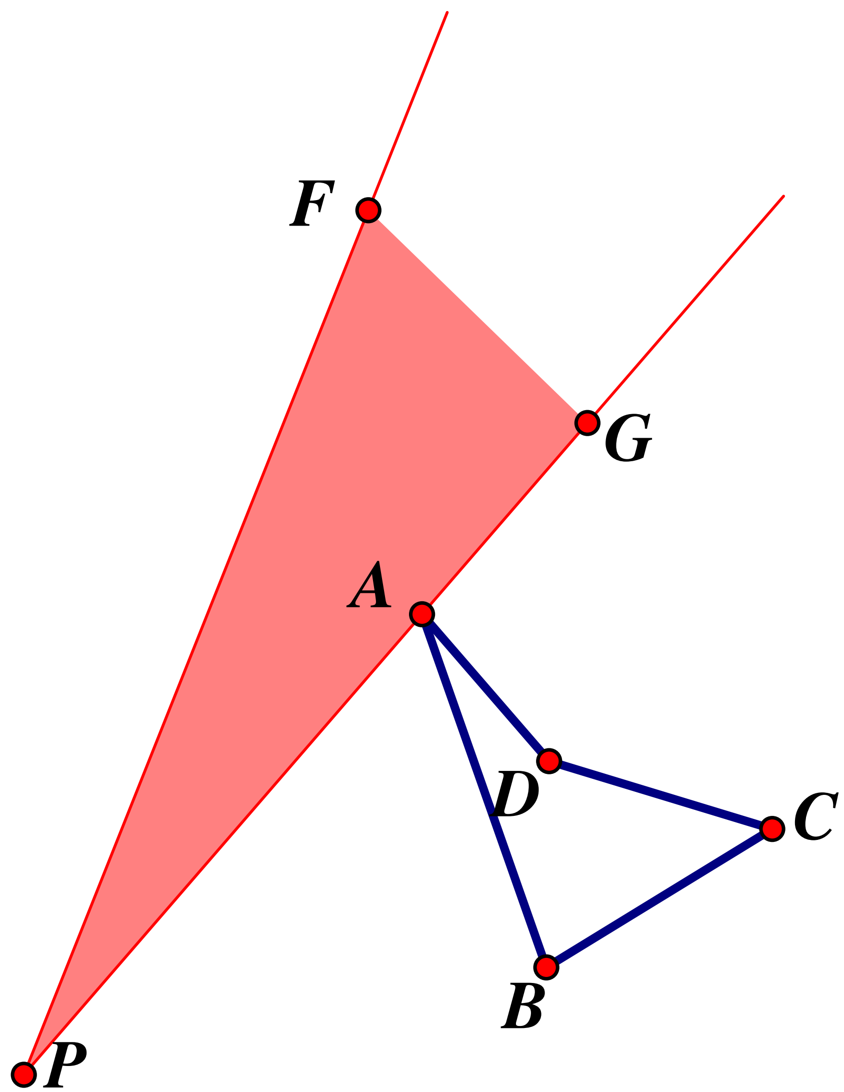
Translating by Tracing

Rotation

- Rotate ABCD with center P an angle GPF. What to do?

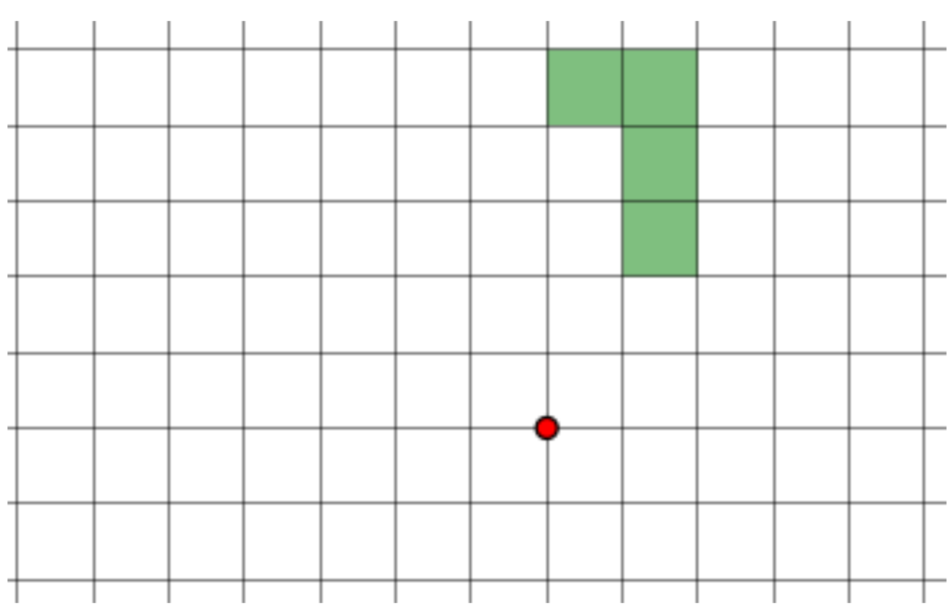
- One idea: Use a wedge of cardstock like the red shape and use it to rotate the rays PA, PB, PC, PD and then mark off the lengths PA, PB, PC, PD to get the rotated shape.

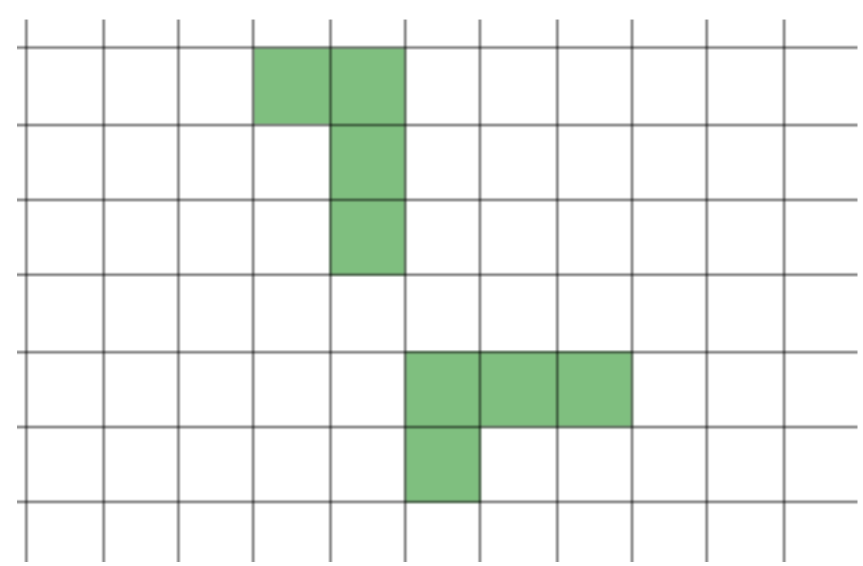
- Second idea: reflect ABCD in the line PA and then reflect again in the angle bisector of GPF. Does this work?



The Coordinate Plane

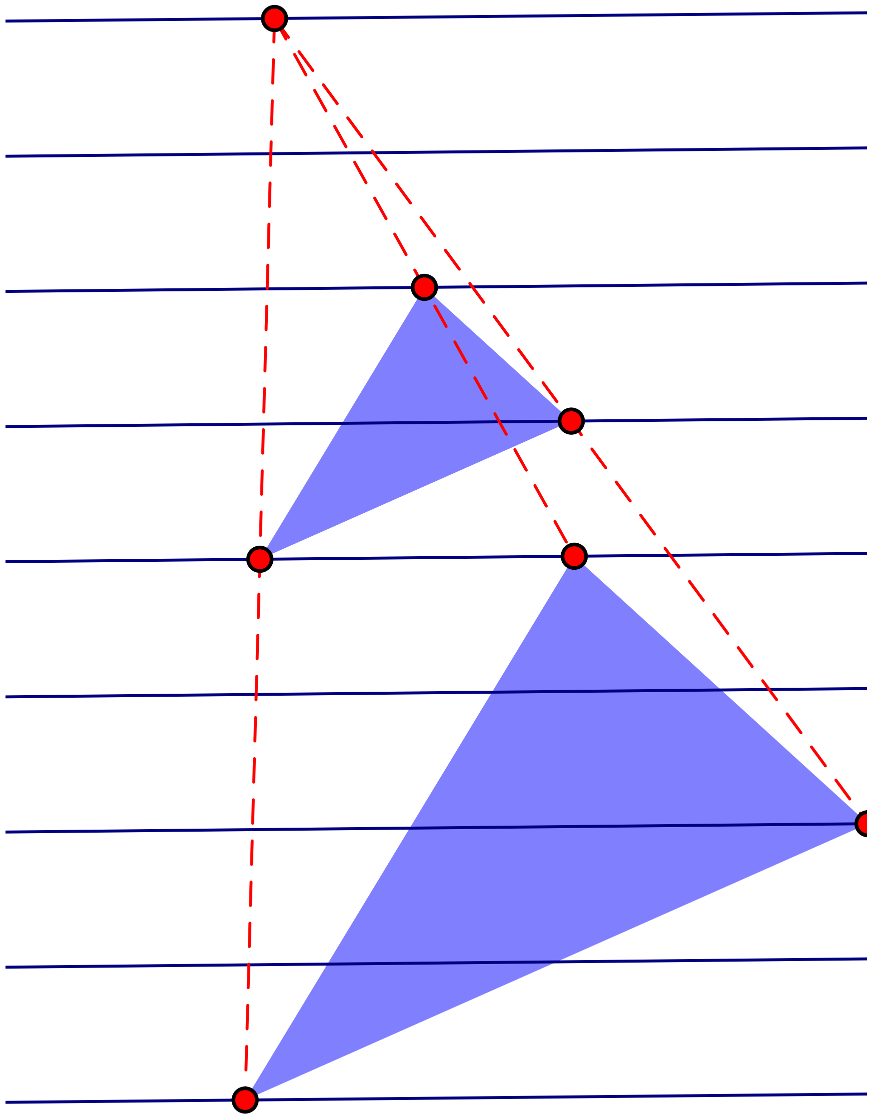
- Graph paper!
- Examples: In the left figure, rotate the shape by 90 degrees with the center point shown.
- In the right figure, find the center and angle of rotation that takes one shape to the other.





Dilation by notebook paper

- Euclid did not have notebook paper. You have lots.
- So you can dilate a shape like this. Just count the spaces between the line and take the ratio to see the ratio of dilation.



Dilation by dot paper

- Lots of similar triangles in this paper. Look for a dilation? Find the center.

- Or start with a corner triangle and dilate it to a bigger one.