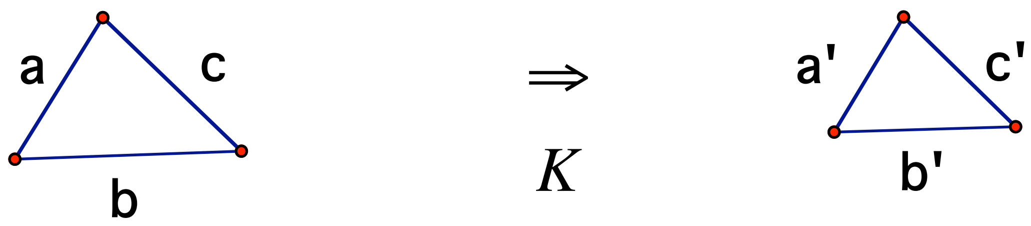
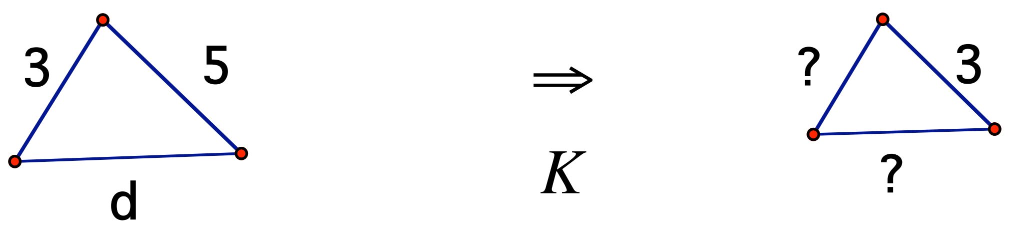
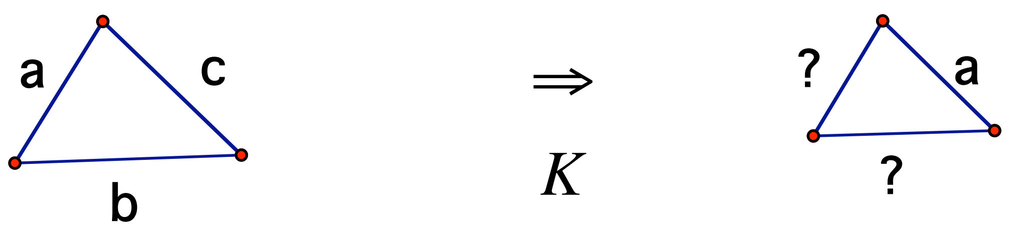
1. A graphical organizer for the data of similar triangles

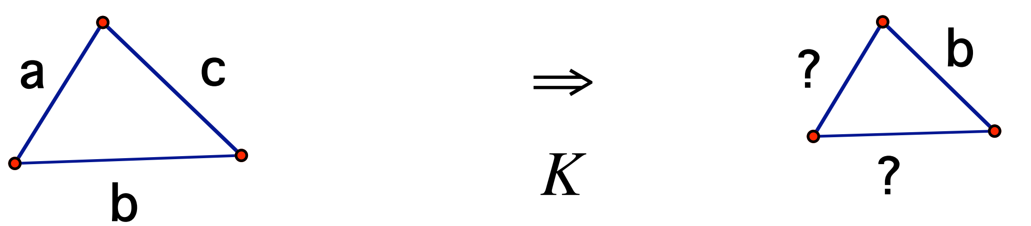
This is the kind of graphical representation that we use for two similar triangles with scaling factor K. In this case, a’ = Ka, etc. We make no attempt to draw the triangles at the correct scale, since this is merely a graphical organizer for the data.



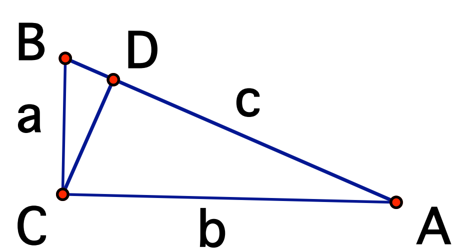
Problem: Fill in the missing values for sides in these diagrams and find the value for K in each case.







Application 1A: In this triangle, angle C is a right angle and CD is perpendicular to AB. The lengths AB = c, BC = a, CA = b. Find the lengths of all the other segments in the figure: AD, BD, CD. **Hence a famous theorem.**



For material from this presentation, go to www.math.washington.edu/~king (after 5/1)

2. Dilations with positive and negative ratios

A dilation of the plane with center A and non-zero ratio K is a transformation that takes a point P to the point P' on line AP so that AP'/AP = K.

There are two important cases:

- If K > 0, then P' is a point on ray AP.
- If K < 0, the P' is the point on the ray opposite AP with |AP'| = K|AP|. (|AP| denotes length.)

Among the properties that can be proved about such a dilation are that for any P and Q, (1) if P' and Q' are the images by the dilation, then |P'Q'| = |K||PQ|. (2) a dilation maps segments to parallel (or collinear) segments.

Notebook paper examples

Notebook paper is a great tool for constructing dilations with little explicit measurement.

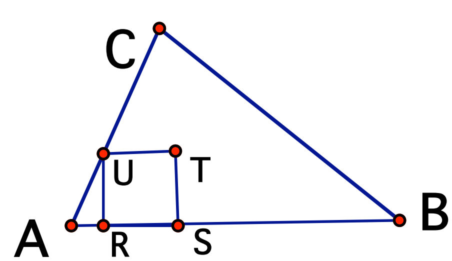
- Draw a segment PQ on one of the lines of the notebook paper and draw a point A also on a line. Then use a straightedge and the spacing of the notebook paper to draw the segment MN that is the dilation of PQ by ratio ½. What is the ratio AM/MP?
- Continue and draw the segment XY that is the dilation of PQ by ratio -1/2.
- What is the ratio AM/MP?
- What is the ratio AX/XP?
- Repeat the exercise for K = 1/3 and -1/3.

3. Some applications of dilations

Example 1. (A problem of Polya) Draw any acute triangle ABC. What is the largest square EFGH that can be constructed inside the triangle so that the side EF is contained in AB and the vertices G and H are both on sides of the triangle.

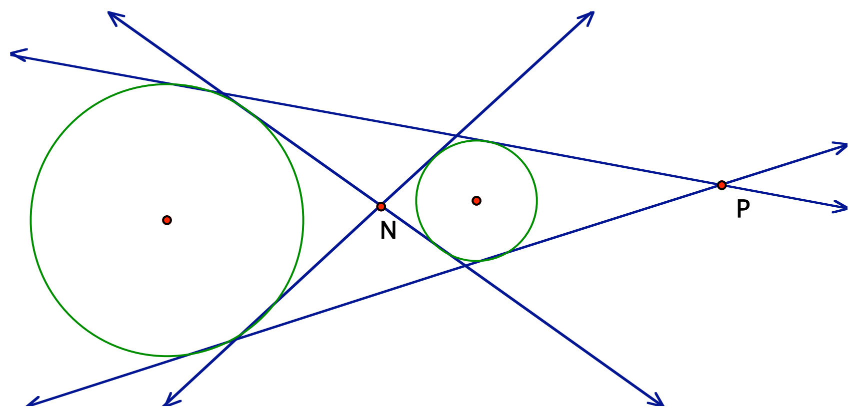
Dilation approach: Draw a triangle RSTU so that RS is in AB and U is on CA. Then dilate with center A to solve the problem.

What ratio will be needed? Where will a vertex of the solution square be located on BC?



Example 2. Common tangents of two circles

Given two circles, how can one construct the common tangle lines? For example, given these two circles, we would like to construct the tangent lines shown here.



The point P is the center of a dilation that takes one of the circles to the other with a positive dilation ratio equal to a ratio of the radii. The point N is the center of a dilation that has a negative dilation ratio. Once the points P and N are constructed, it is only necessary to construct the tangents from the point to one of the circles and the tangent lines will also be tangent to the other circle. (There is some reasoning required to justify this statement.)

So to make this construction, we need to consider how, given two similar figures, we can construct (if possible) one or more centers of dilation that map one figure to the other.

4. Constructing Centers of Dilation

Another Notebook paper exercise
Draw two segments AB and CD on the notebook paper, each segment lying on a different line of the notebook paper (thus the segments will be parallel).

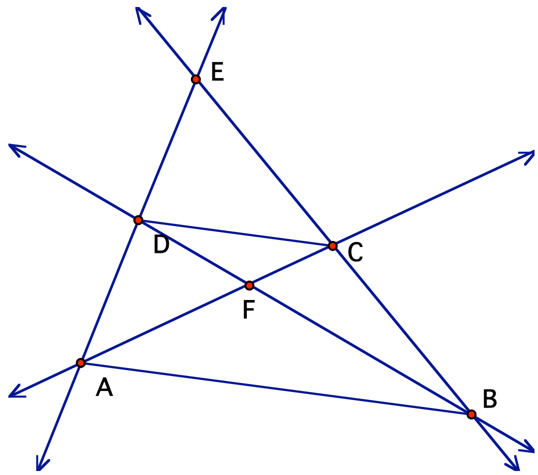
Just using a straightedge, construct a point P so that P is the center of a dilation that takes A to C and B to D?
Just using a straightedge, construct a point Q so that Q is the center of a dilation that takes A to D and B to C?

Questions:

a) What can you say about the signs of the ratios of dilation of these two dilations?
b) Can there be more than two centers of dilation that take segment AB to segment CD?
c) Can there be less than two?
d) What happens if the segments are collinear?

5. Trapezoids, dilations and ratios

In the previous section, you constructed two centers of dilation for any two parallel segments of different length. The picture looks like this.



There is more than one way of thinking of this figure:

- It shows the centers E and F of two dilations taking AB to CD.
- It is a trapezoid ABCD with two diagonals intersecting at F and two non-parallel sides intersecting at E.
- It is a triangle ABE, with segment from side AE to side BE parallel to AB. The lines AC and BD intersect at F.

It is possible to use the dilations to answer questions about ratios in the figure, and hence about ratios in trapezoids and triangles.

Question 1. Draw the line EF, intersection AB in P and CD in Q. Explain why P is the midpoint of AB and Q is the midpoint of CD. (Hint: Why do dilations map midpoints to midpoints?)

Question 2. Suppose that |CD| = (1/2)|AB|. (You may want to draw this case accurately on notebook paper.

Find the following ratios:

- ED/EA
- FA/AC
- FP/EP (This is a famous theorem.)

Question 3. Suppose for the triangle ABE in this figure that ED/EA = EC/EB = 1/3. What is the ratio PF/PE?

Question 4. Suppose for the triangle ABE in this figure that ED/EA = EC/EB = 2/3. What is the ratio PF/PE?

Question 5. In each of these two previous cases, what is the ratio of the areas of triangle ABF and ABE?

6. Affine Geometry Theorems

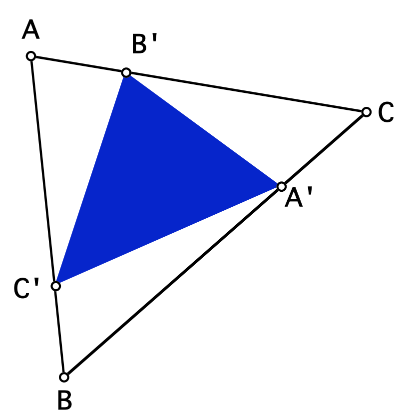
Think about the results of the last section about (signed) ratios of lengths and ratios of areas. Notice that these were true for any triangles, not just special triangles.

These theorems are example of affine plane geometry. This geometry uses the usual plane but does not use distance measure or angle measure. An affine transformation is a transformation that may not preserve angles or distances but does map parallel lines to lines that remain parallel to each other (though not necessarily to the original lines, as is true of dilations). Two figures are affine-congruent if one can be mapped to the other by an affine transformation. These transformations also preserve ratios on lines but not necessarily ratios on lines that are not parallel.

In affine geometry (1) all triangles are affine congruent; (2) all rectangles and parallelograms are congruent. (But not all quadrilaterals are affine congruent.); (3) circles and ellipses are congruent; (4) not all trapezoids are congruent, since the ratio of the parallel sides is defined in affine geometry.

IF THIS ALL SEEMS TOO ABSTRACT, an example of an affine transformation is given by dragging the vertex of a triangle using dynamic geometry software.

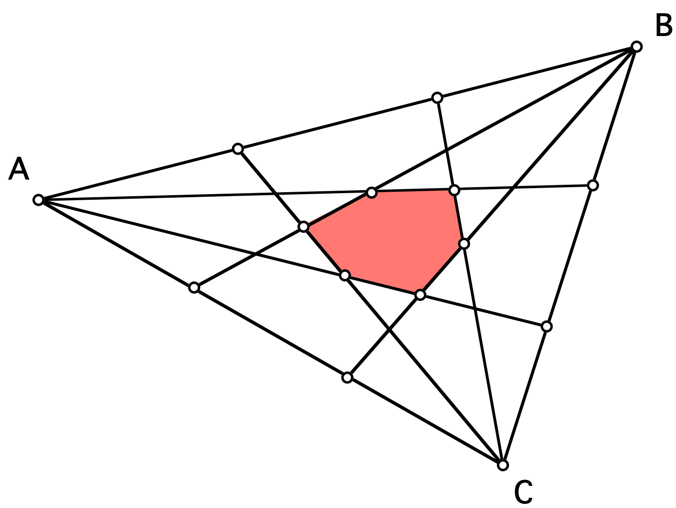
Example 1: Midpoints and medians make sense in affine geometry. The concurrence of the medians in a triangle is an affine geometry theorem.
Example 2: Perpendicular bisectors are not defined in affine geometry, since angle measure is not defined.
Example 3. The theorem that for a trapezoid, the line through the intersection of the diagonals and the intersection of two non-parallel sides passes through the midpoints of the parallel sides is an affine theorem.
Example 4: In this figure, suppose each of the points A'. B', C' divides a side of triangle ABC, all in the same ratio. Then ABC is divided into 4 sub-triangles, as in the figure. The areas of the unshaded sub-triangles are all equal. For each sub-triangle, the ratio of the area of this triangle to the area of ABC only depends on the ratio r = AC'/AB = BA'/BC = CB'/CA, not on the shape of the triangle. (What are the ratios?)



7. Marion Walters' Theorem

This is a beautiful example of an affine theorem. It is not clear who actually first proved this theorem, but it was studied in papers by Marion Walter's and her name is as a consequence attached to the theorem.

The theorem is true for any triangle, if are divided into thirds and segments are drawn as in the figure. The theorem states the ratio of the area of the central shaded hexagon to the area of ABC. (We are not stating this ratio yet to make the problem more fun to solve.)



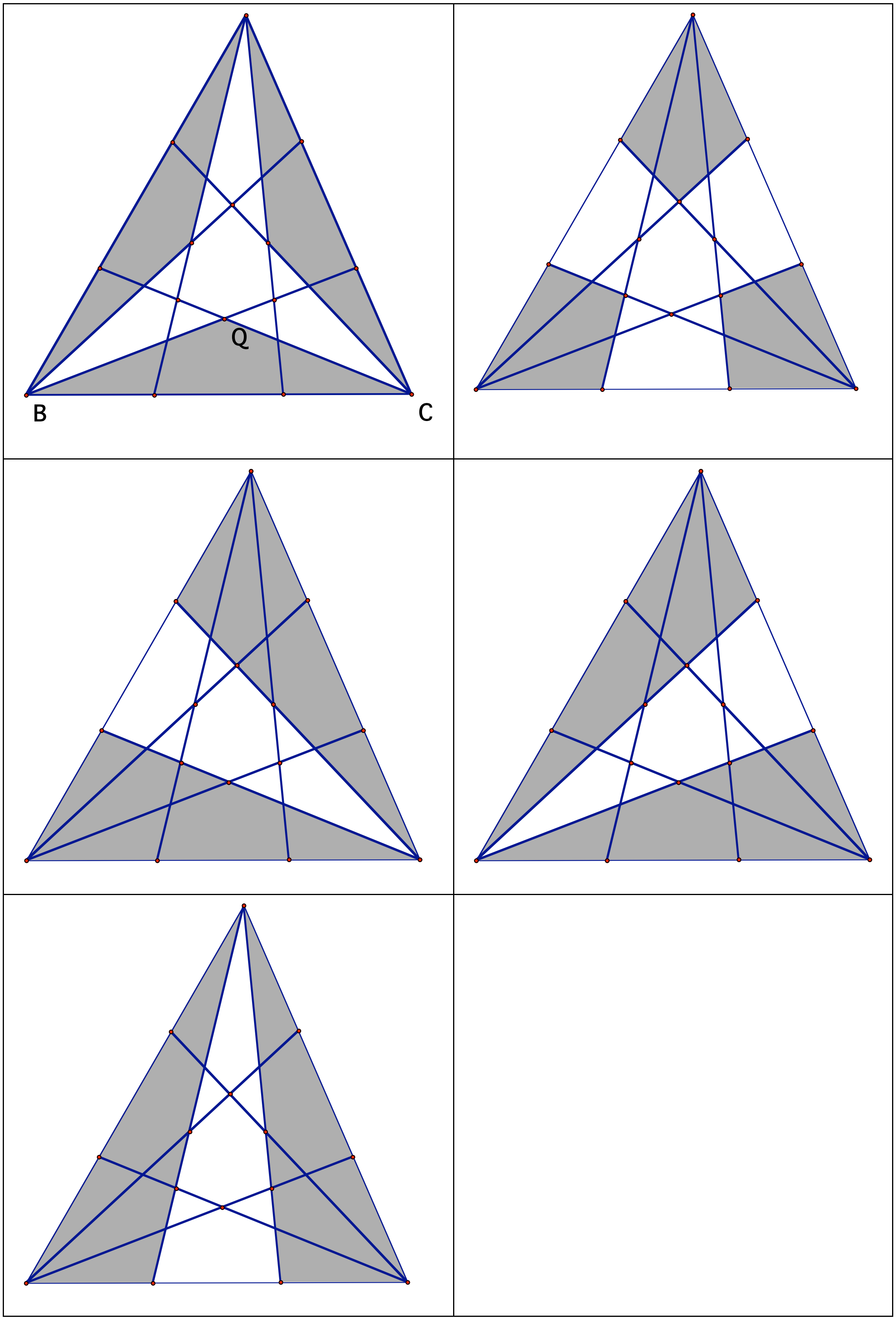
A general approach to the solution would be to find (as a ratio times the area S of ABC) the areas of various pieces in the figure. Given the limited time for this workshop, we are going to suggest some pieces on the next page. Some of these you have already computed earlier.

Since this is an affine theorem, we can prove it for any triangle. The figures on the net page are actually the case of an equilateral triangle, but the reasoning works for any triangle.

In each case, assuming the area of ABC is S, what number times S is equal to the area for each of the 5 cases on the next page (some are really the same).

Once you have the areas, add up the last three areas. This covers the complement of the hexagon, but does it more than once for some of the polygons. So subtract the areas of the first two figures and think why the result is exactly the area of the complement of hexagon. Hence the area of the hexagon can be computed.

Final Example: Ceva's Theorem. Probably no time for this, so you can look it up if we do not discuss this theorem.


Q
B
C