BASING YOUR GEOMETRY COURSE ON RIGID MOTIONS AND DILATIONS

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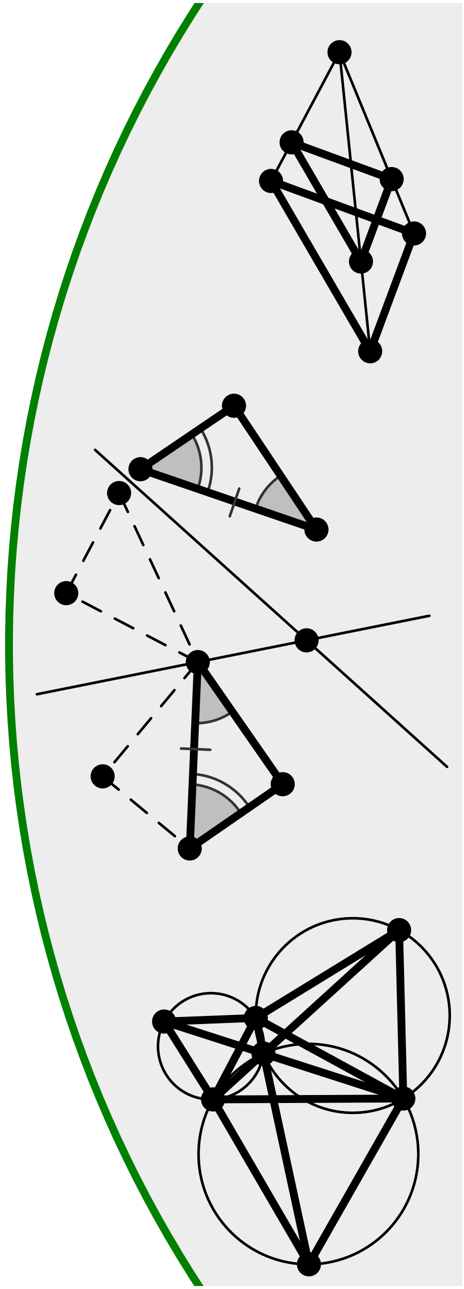


TRANSFORMATIONS IN GEOMETRY CLASS FROM THE GET-GO

We can introduce transformations–rigid motions and dilations–at the foundational stages of a geometry course.

It is not as challenging as one might think; sometimes it is easier than familiar ways.

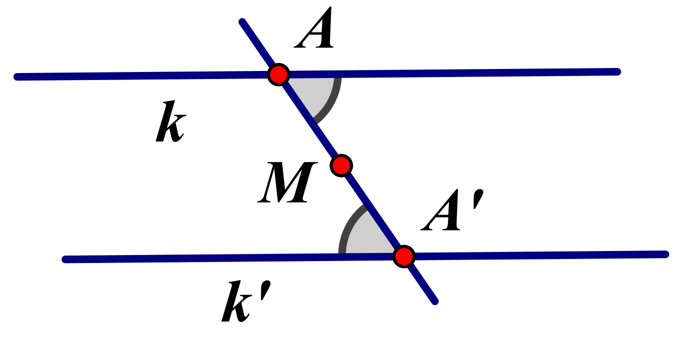
Big payoff: You have everything you had before plus a powerful new tool ready to use.

OUTLINE OF THIS TALK

1. Previews of thinking with transformations
2. Rigid motions, congruence and triangles
3. Dilation Axiom and Euclidean Parallel Postulate.
4. Similitudes and similarity in general.
5. Problem-solving with dilations.

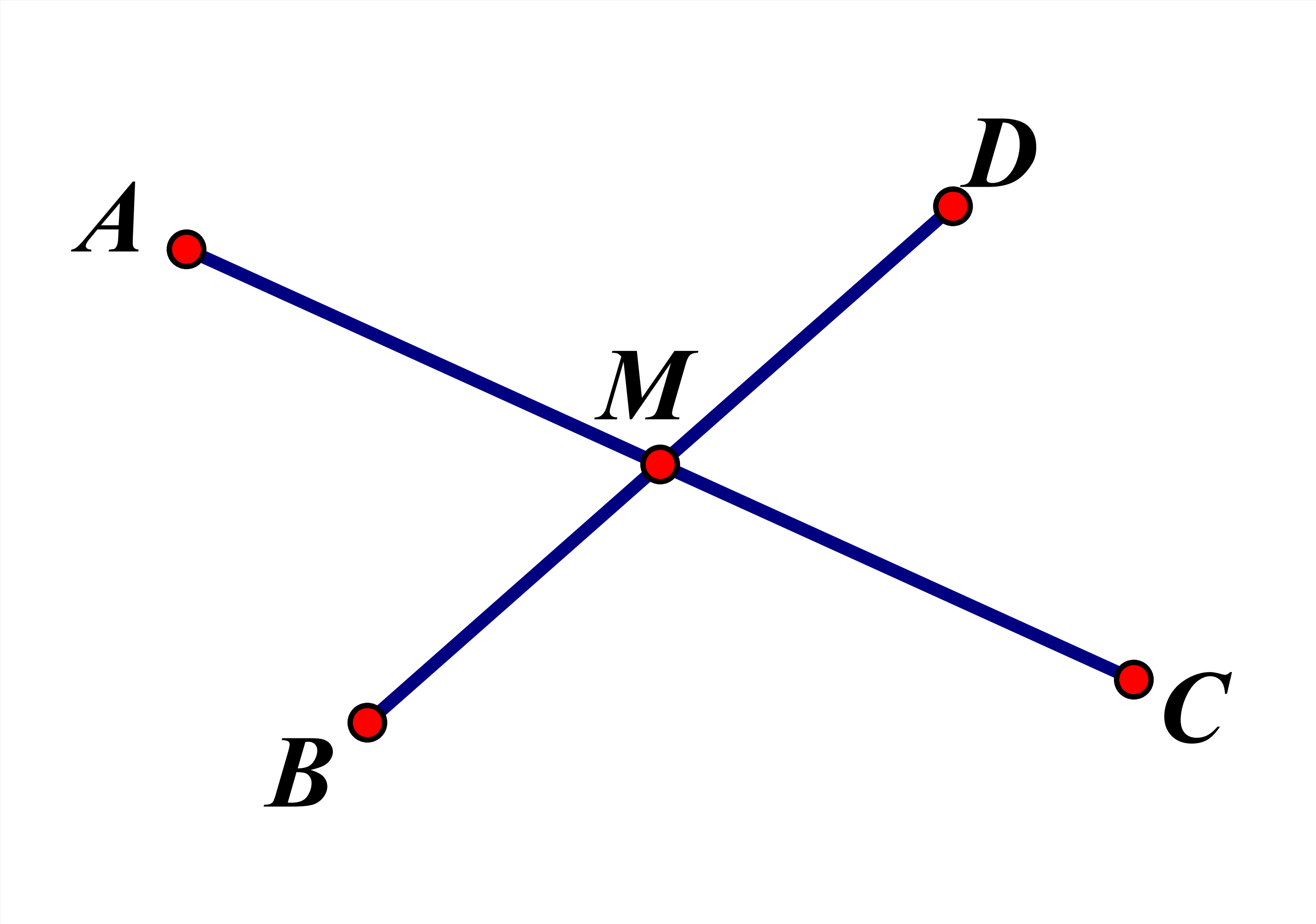
PREVIEW: HALF-TURN SYMMETRY

- A **half-turn** is a rotation of 180 degrees. If H is a half-turn, for any A, let A' = H(A).
- The center M is the midpoint of AA' and H(A') = A.
- The image of a line k not through M is a line k' parallel to k.
- A figure F has **half-turn or point symmetry** if H(F) = F,
- Angles related by H are congruent, as in this familiar figure with point symmetry.



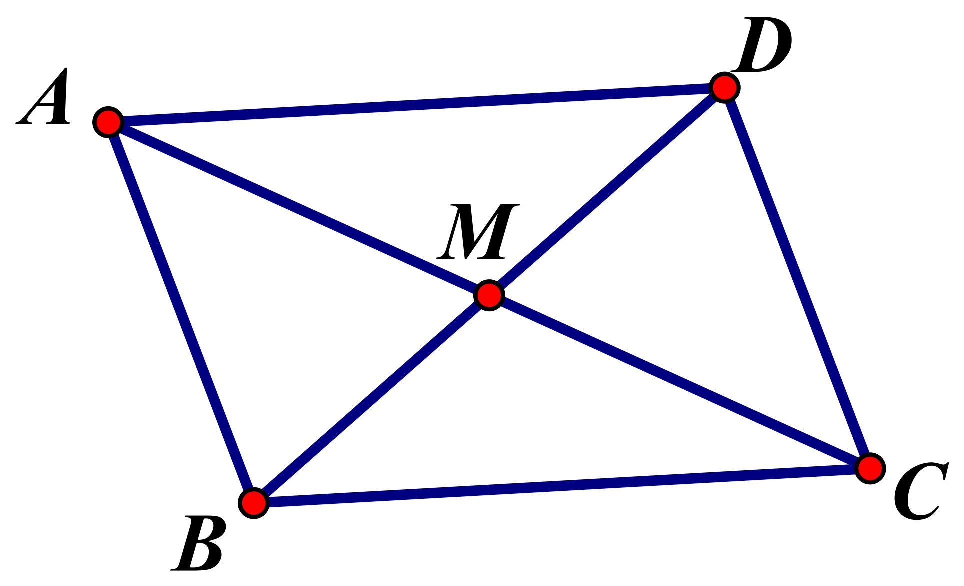
PREVIEW: PARALLELOGRAMS

- Suppose points ABCD have a half-turn symmetry.
- There is a half-turn H with H(A)=C and H(B) = D, with center M the midpoint of both AC and BD.



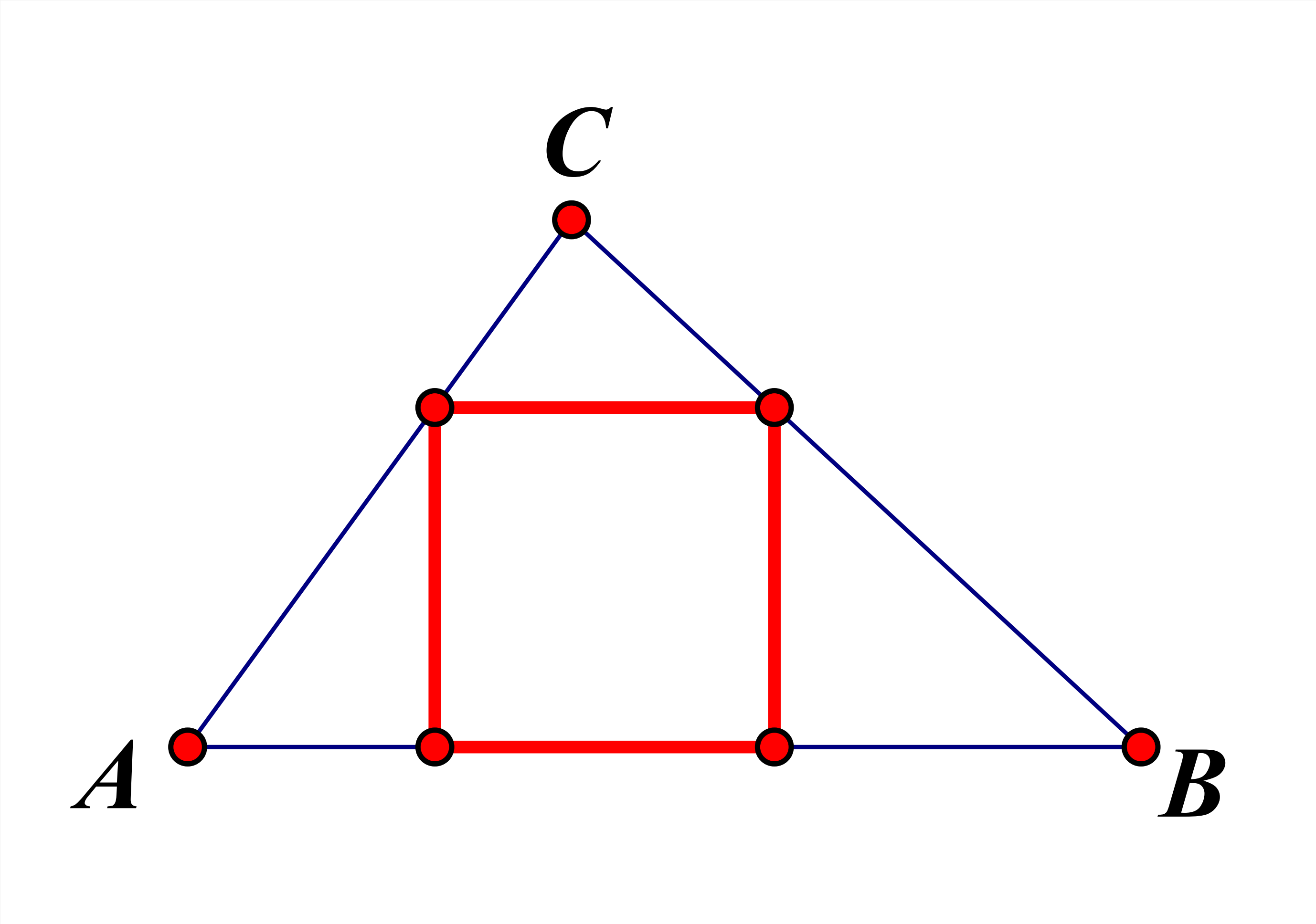
PREVIEW: PARALLELOGRAMS

- Then the opposite sides of ABCD must be parallel and congruent. Converse also true.
- In fact, parallelograms can be defined as quadrilaterals with half-turn symmetry.
- All the usual properties follow immediately.



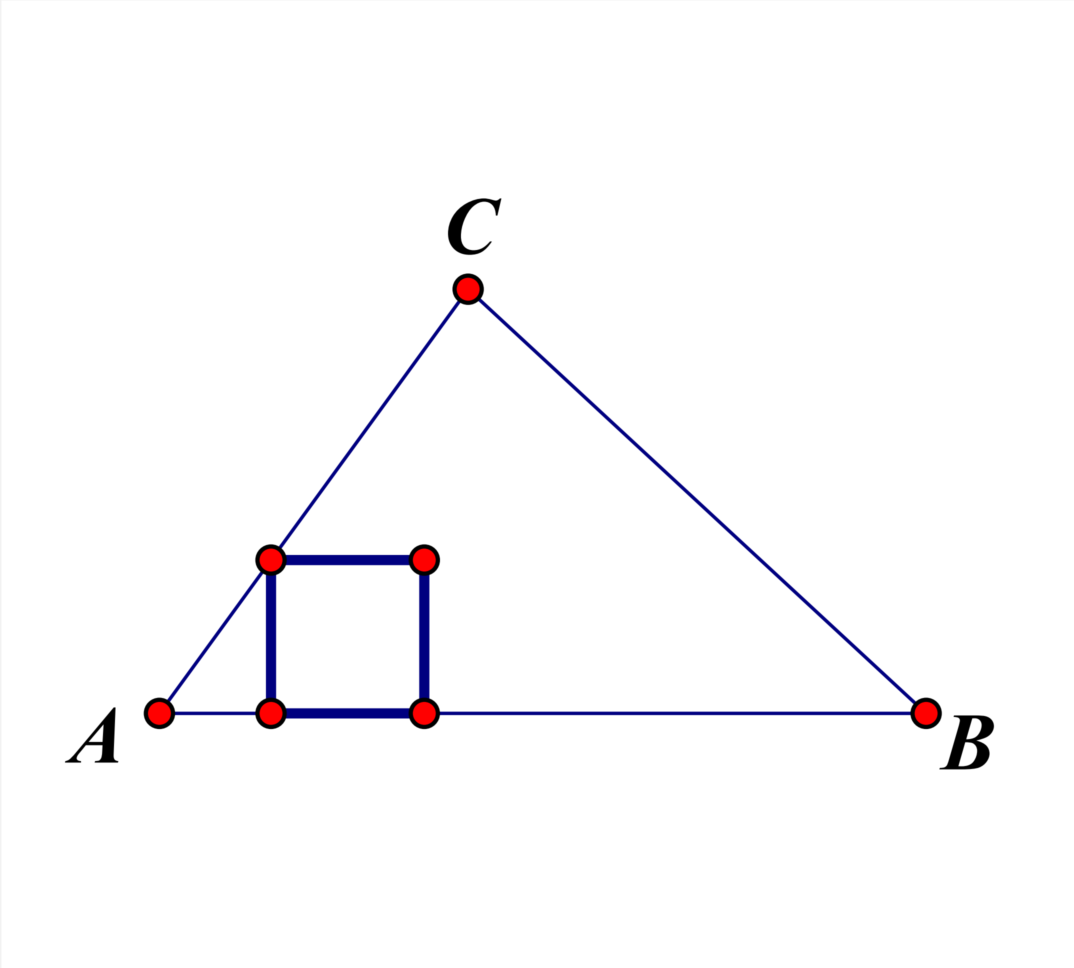
PREVIEW: POLYA SQUARE IN TRIANGLE

- Given a triangle ABC.
- Polya poses the problem of constructing a square inscribed in the triangle, as shown in this figure.



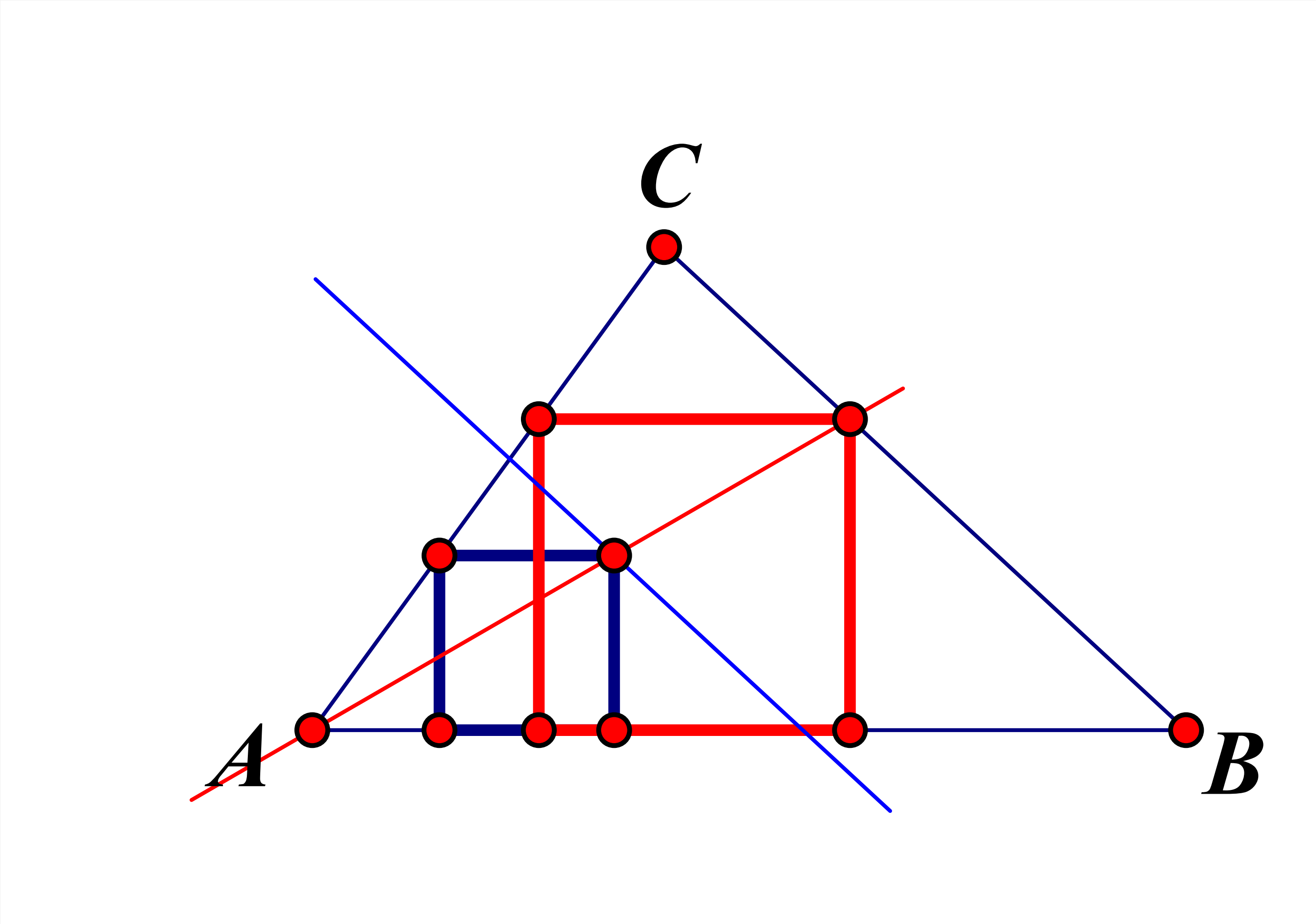
PREVIEW: POLYA SQUARE IN TRIANGLE

- Polya's solution is to partially solve the problem by constructing a square that touches at 3 vertices.
- This small square is a scale model of the desired figure, if we add a line parallel to BC.



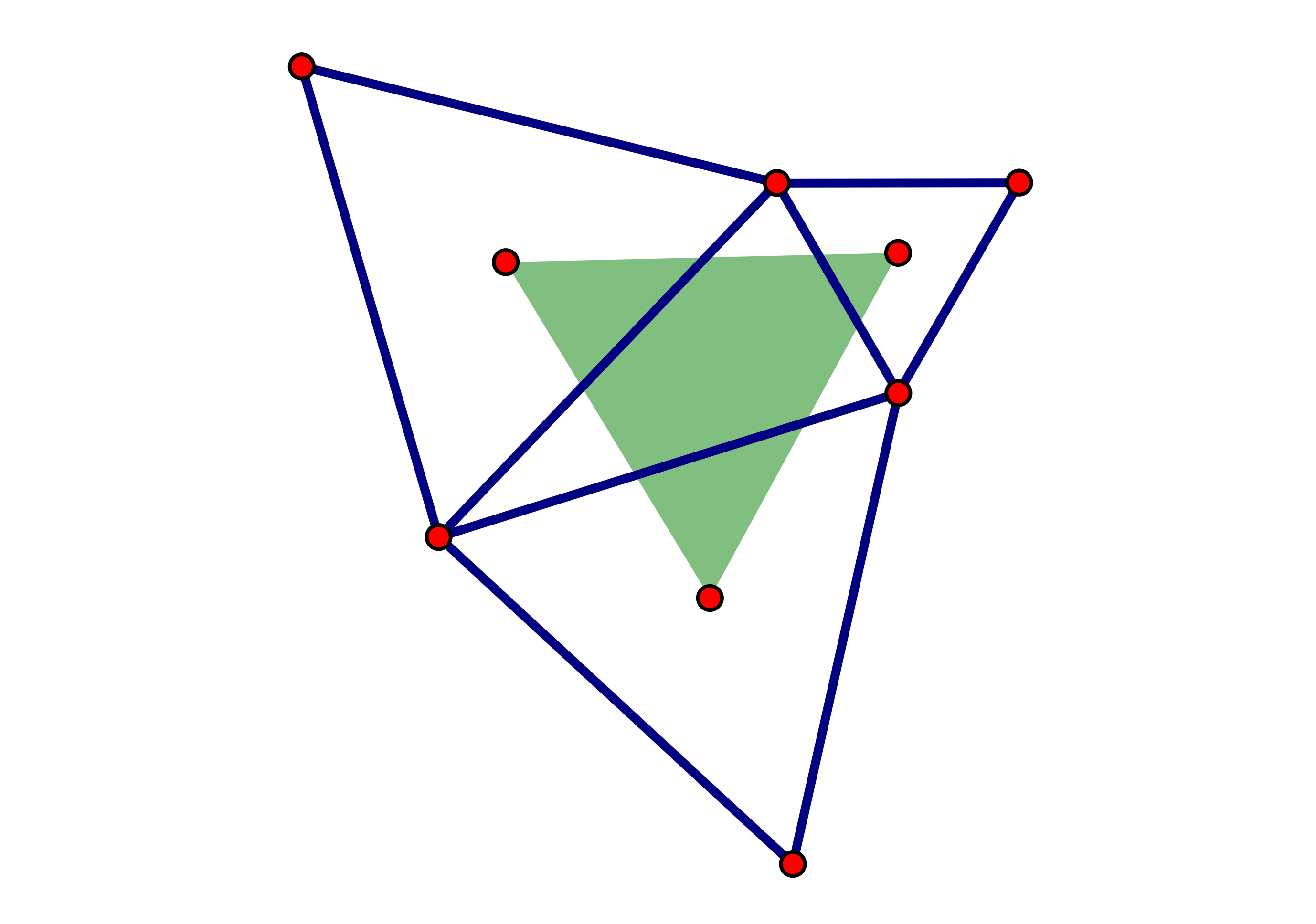
PREVIEW: POLYA SQUARE IN TRIANGLE

- Then we can dilate with center A so that the image square has a vertex on BC.
- This image point is constructed by intersecting BC with the red line through A and the vertex of the blue square not on a side of ABC,



PREVIEW: NAPOLEON'S THEOREM

- The preceding examples have nice proofs without transformations. This one is hard without rotations.
- Construct 3 equilateral triangles on the sides of any triangle. Then the centers of these triangles form an equilateral triangle



STARTING POINT: BASIC AXIOMS

1. The plane has **POINTS** and **LINES**; any two distinct points A and B lie on a unique line.
2. The plane is provided with a **DISTANCE** measure $0 \leq |AB|$ between any points A and B. With this measure lines look like the real number line with its usual distance.
3. The plane is provided with an **ANGLE MEASURE** $0 \leq \measuredangle ABC \leq 180$ for points B and C distinct from A -- with the usual properties.
4. To establish that the plane is 2D and not 3D, we assume that the complement of a line is two **HALF-PLANES** with the usual properties.

RIGID MOTIONS: TRANSFORMATIONS THAT PRESERVE DISTANCE AND ANGLE MEASURE

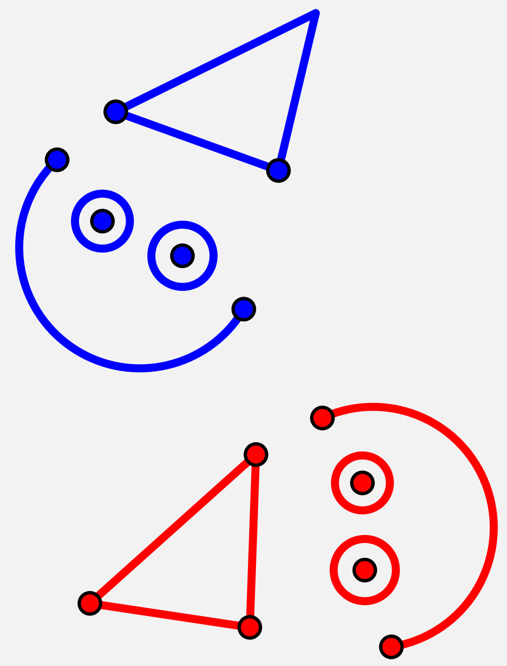
- If T is a rigid motion, always $|T(A)T(B)| = |AB|$.
- If T is a rigid motion, always $\measuredangle T(A)T(B)T(C) = \measuredangle ABC$.
- Rigid motions map lines to lines, since distinct points A, B, C are collinear if and only if $\measuredangle ABC = 0$ or 180.
- So, a rigid motion is a map that preserves all the geometric properties in the axioms.

DEFINING CONGRUENCE USING RIGID MOTIONS

- **Definition: A set F is CONGRUENT to G there is a rigid motion T that maps F onto G.**
- **In this case, G is also congruent to F, since the inverse T^{-1} maps G to F (T^{-1} is also a rigid motion).**
- Informally, two figures are congruent if you can **superimpose** one on the other. Greek geometers of antiquity had this intuitive idea too but had no math language for it. But we have the language of **functions**.

ADVANTAGES OF THIS DEFINITION VS. SAS TRIANGLE CONGRUENCE

- Defines congruence as a concept for ***any*** pair of figures, such as two circles or ellipses. Also, lines, or disconnected figures!
- One catch: To make this useful, we need a supply of rigid motions



HOW TO GET RIGID MOTIONS

- To get a supply of rigid motions, we start with some simple types and then get more by composing them.
- **Axiom (less visual form):**
- **For any line m, there is a rigid motion, distinct from the identity map, that fixes the points of m.**
- **Note:** If a rigid motion fixes points A and B, it fixes all the points of line AB, because of distance.

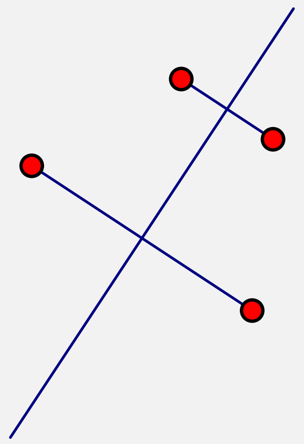
LINE REFLECTION & AN AXIOM WITH MORE CONCRETE WORDING

Suppose a transformation T fixes the points of m; and for any point C not on m, the line m is the perpendicular bisector of CT(C).

T is called **line reflection in m**, denoted R_m.

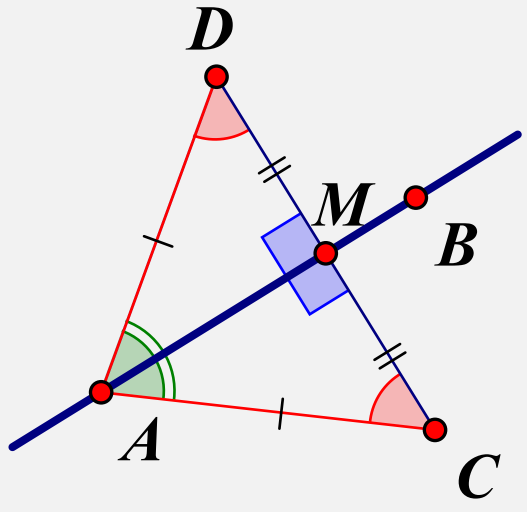
Axiom (more visual version):

For every line m, the line reflection in m exists and is a rigid motion.



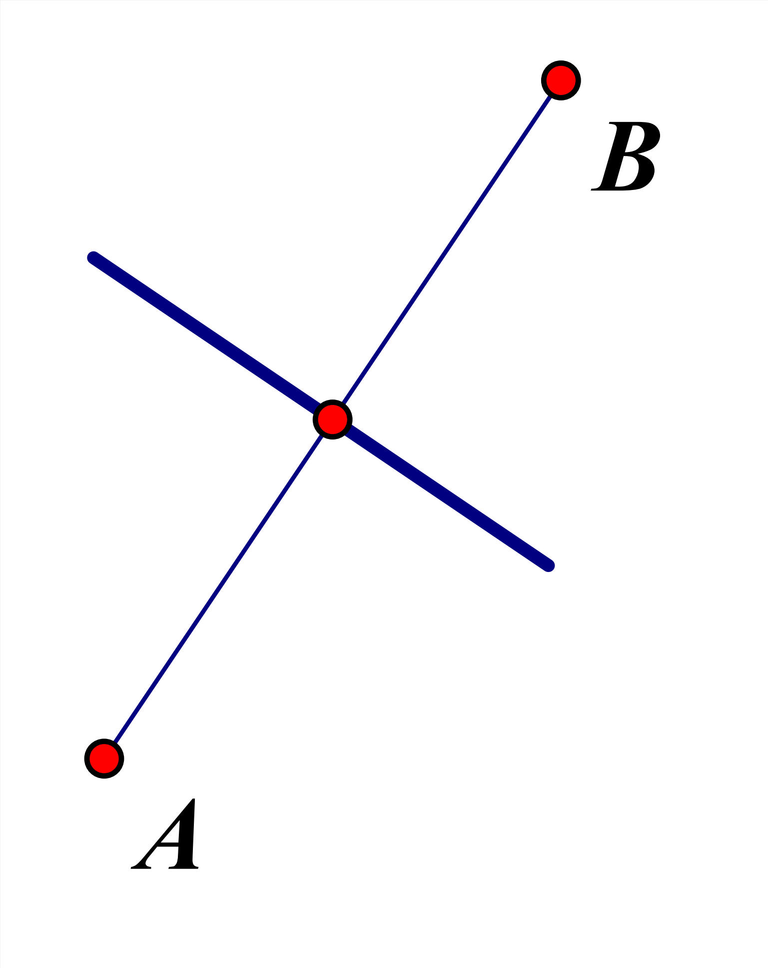
THESE AXIOMS ARE EQUIVALENT

- Assume T fixes A and B and T(C) ≠ C. If we connect points with line segments, we see relationships if T is a rigid motion.
- (a) There is only one possible value D for T(C) because ∡CAB = ∡DAB and |DA| = |DC|.
- (b) T(MC) = MD since M is on the line AB.
- (c) All the pairs of marked angles and segments are congruent! This contains isosceles triangle properties!
- (d) Since line AB is the perpendicular bisector of CD, **T is reflection in line AB**



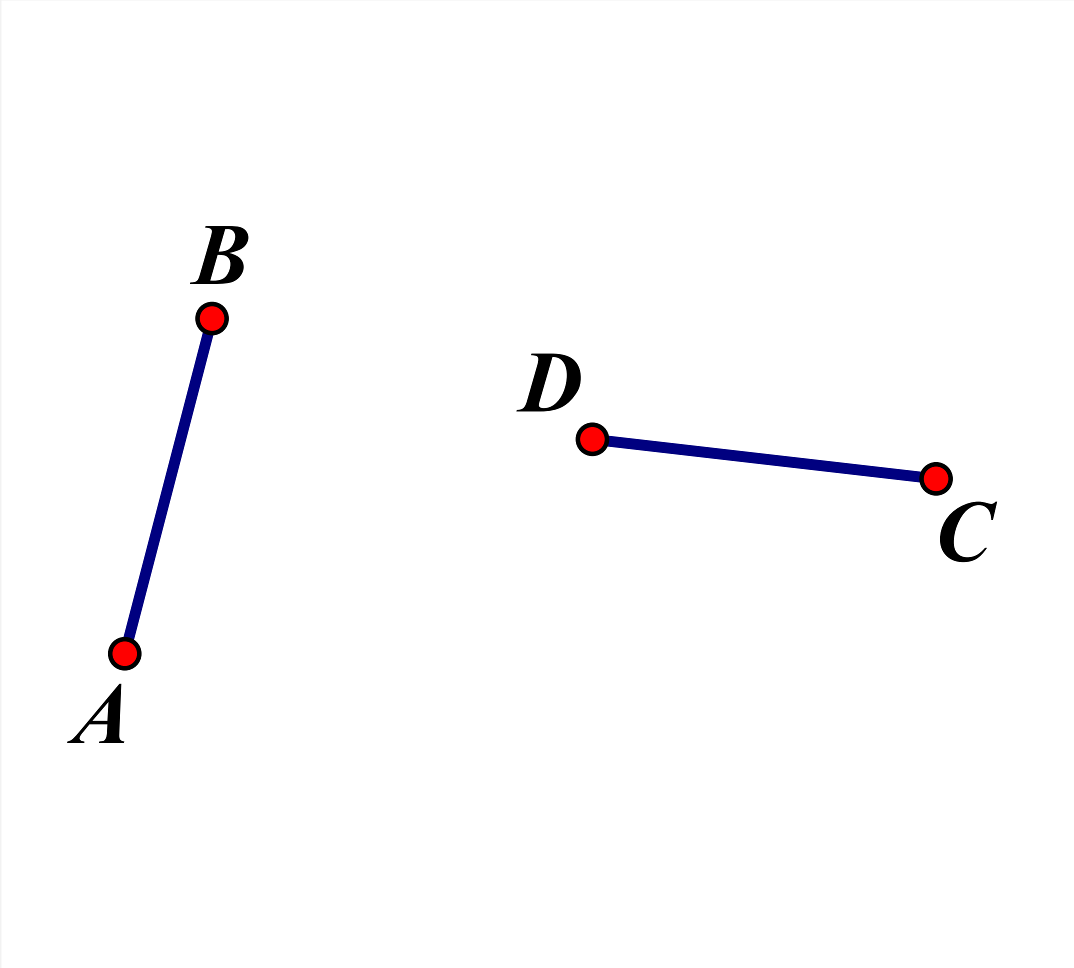
BASIC CONGRUENCE THEOREMS

- **Case 0: Any two points A and B are congruent.**
- Yes, we need to prove this using the definition! We must find a rigid motion that takes A to B.
- Reflection in the perpendicular bisector works!
- What rigid motion will you use if A = B?



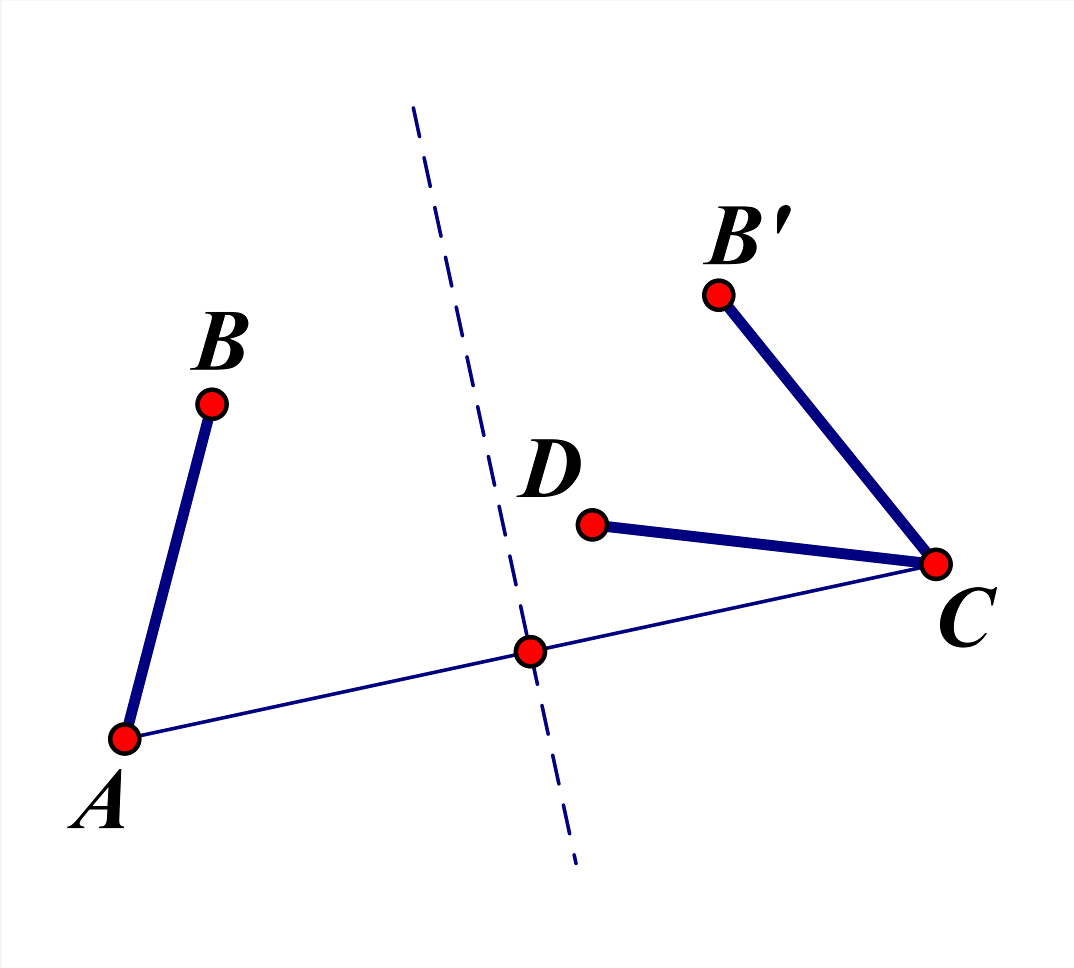
CONGRUENCE OF SEGMENTS

- Suppose AB and CD are segments of equal length.
- Can we find a rigid motion that takes AB to CD?
- We just learned how to map A to C.



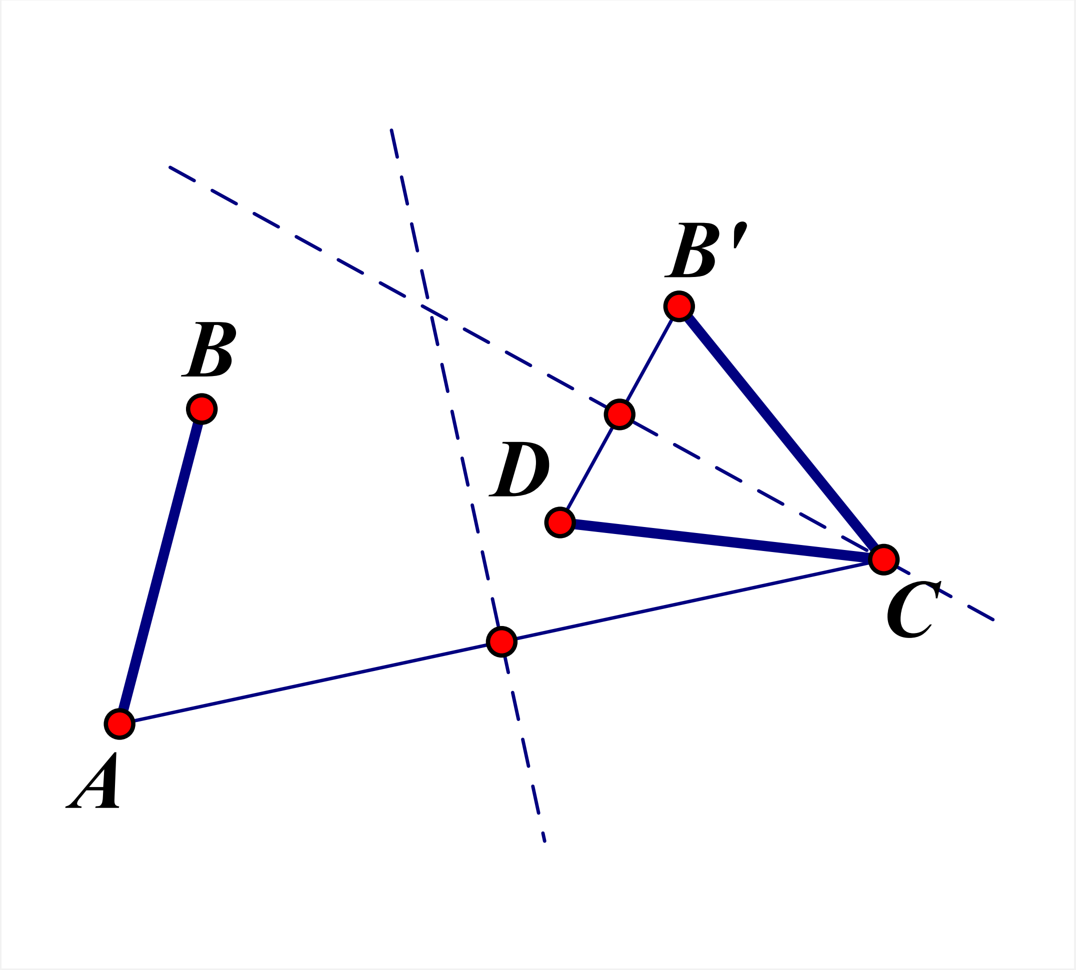
MOVING THINGS: SEGMENTS

- If A = C, triangle ABD is isosceles. Go directly to step 2.
- **Step 1**. Assuming A≠C, reflect AB in the perpendicular bisector of AC.
- The image of AB is now CB', with |B'C| = |DC|. If B' = D, we are done; stop here.



ANY TWO SEGMENTS OF THE SAME LENGTH ARE CONGRUENT!

- **Step 2.** Triangle CDB is isosceles. Reflect segment CB' in the angle bisector of angle DCB'. CB' reflects to CD.
- Thus, there is a rigid motion that is either one line reflection or the product of two reflections that maps AB to CD.



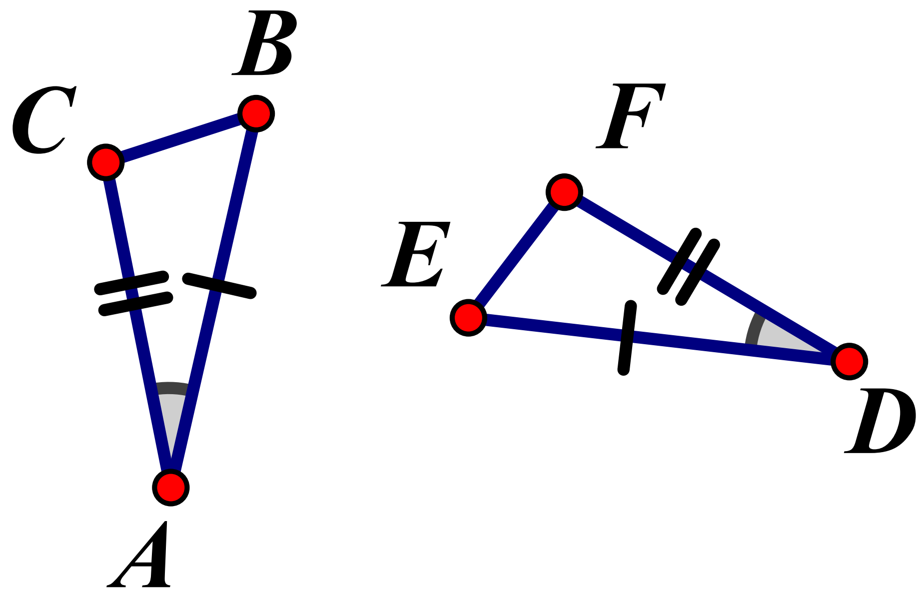
TRIANGLE CONGRUENCE TESTS

Side-Angle-Side (SAS):

Let ABC and DEF be triangles with |AB|= |DE| and |AC|=|DF| and ∡BAC= ∡EDF. Then the triangles are congruent.

Start by mapping AB to DE, then construct one more reflection if needed.

The other triangle congruence tests, ASA and SSS, follow the same pattern..



CONGRUENCE COMMENTS

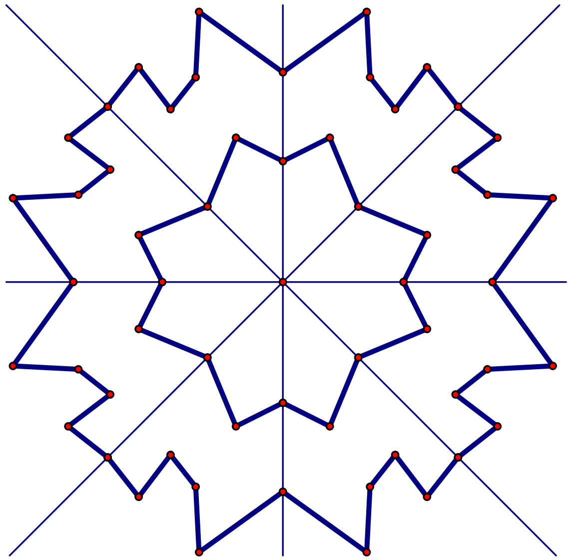
- With SAS proved, we are back in the familiar setup for plane geometry.
- For any triangles ABC and DEF, there is at most one rigid motion that takes A, B, C, to D, E, F; for a rigid motion with three fixed points is the identity.
- SSS for triangles implies that a transformation that preserves distances (an isometry) is also a rigid motion. Important: We can't start with isometries in the Axiom. Need those angles to get us started.
- Looking ahead for more advanced geometry, the rigid motion congruence definition extends to other geometries, such 3-space (or space-time) or spherical geometry.

ABOUT THOSE OTHER RIGID MOTIONS

In addition to line reflections, what are other Euclidean plane rigid motions?

1. The composition of reflections in two lines is a **rotation** with center at the point of intersection if the lines intersect and is a **translation** if they do not intersect. (preserving orientation)
2. The composition of three line-reflections is a glide reflection, or a line reflection in some cases. (reversing orientation)

- **Example**: The figure has 4 line-reflection symmetries. The 4 rotations that are symmetries are products of pairs of line reflections.



SIMILARITY

With what we have so far, we can prove a lot. But we cannot prove some other things, for example:

- Sum of the angles of a triangle = 180 degrees.
- Pythagorean Theorem

This is because, so far, we could be in the non-Euclidean plane. To be Euclidean, we need the Euclidean Parallel Property or the equivalent: Our equivalent will be similarity via the Dilation Axiom.

TRANSFORMATIONS AND SIMILARITY

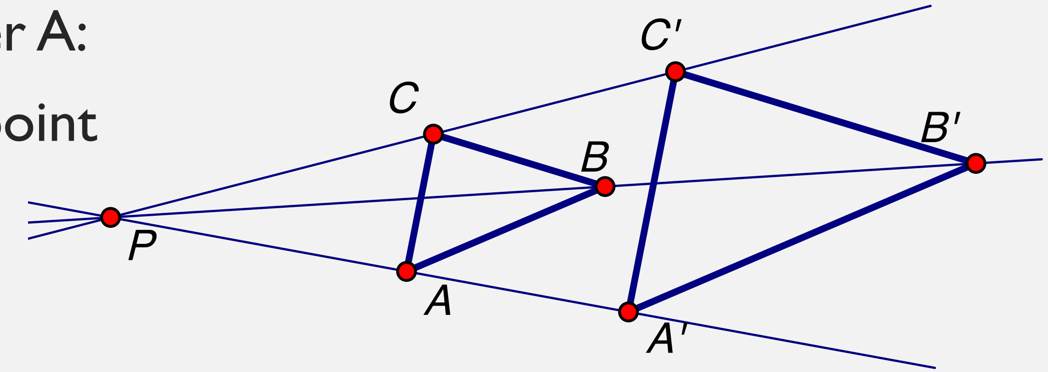
- A ***Similarity Transformation*** with scaling factor $k>0$ is a transformation T that

1. Preserves angles
2. Scales distance by k, i.e., for any A, B, $|T(A)T(B)| = k|AB|$.

- Two figures F and G are ***similar*** if there is a similarity transformation T than maps F onto G.
- Note: T^{-1} is also a similarity transformation with scaling factor $1/k$.
- Note: Rigid motions are similarity transformations with $k = 1$.

DILATION: DEFINITION, THEN AXIOM

- The **dilation** $D_{P,k}$ with center P and ratio k>0 maps P to P ; for other A:
- $D_{P,k}$ (X) = X' where X' is the point on ray PX with |PX'| = k|PX|.
- In the figure, k = 7/4.



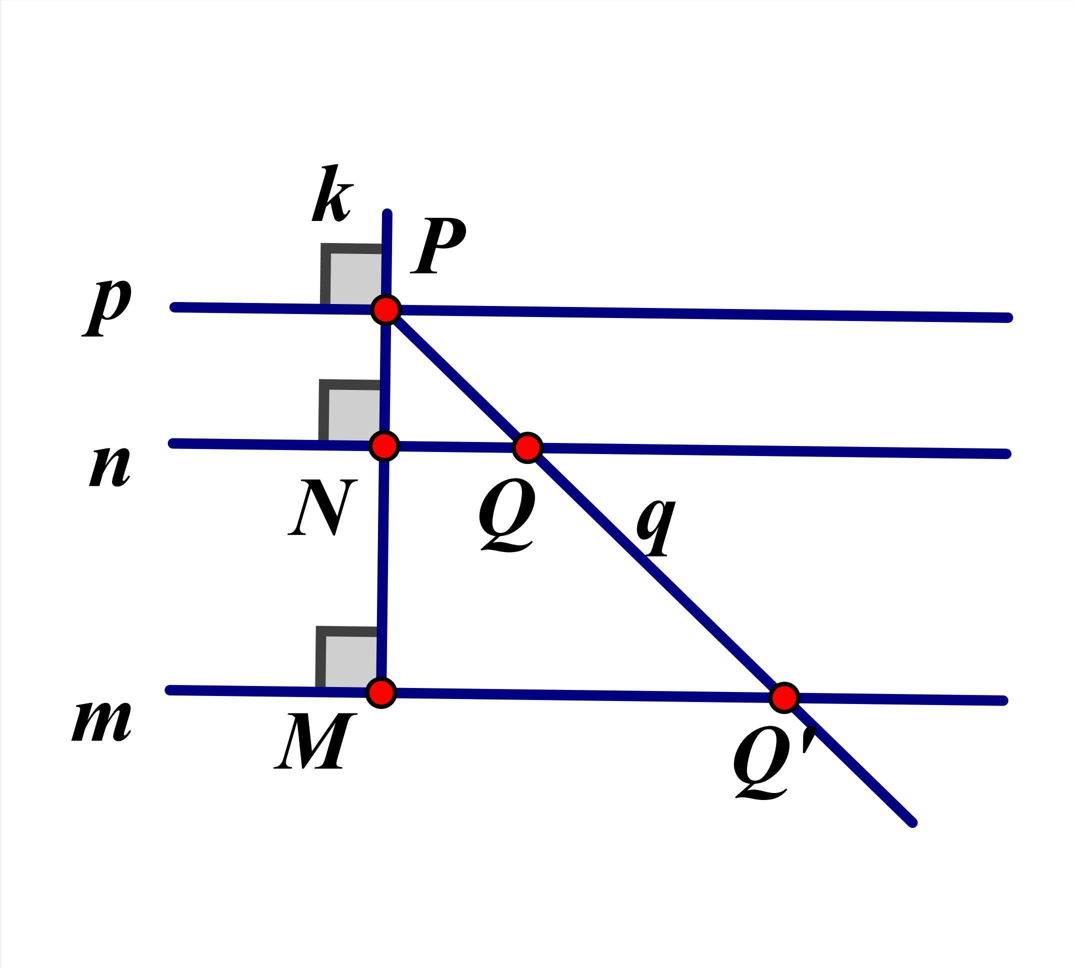
Dilation Axiom: Every dilation is a similarity transformation.

TRANSFORMATIONS AND SIMILARITY AND EUCLIDEAN PARALLEL POSTULATE

- The existence of similar figures in the plane at different scale is equivalent to the Euclidean Parallel Postulate. It implies flatness; try this on a sphere!
- **EPP: For any line m and point P not on m, there is exactly one line through P that is parallel to m.**
- Assuming the DILATION AXIOM, we can prove EPP as a theorem.
- From this rest of Euclidean Plane Geometry follows.

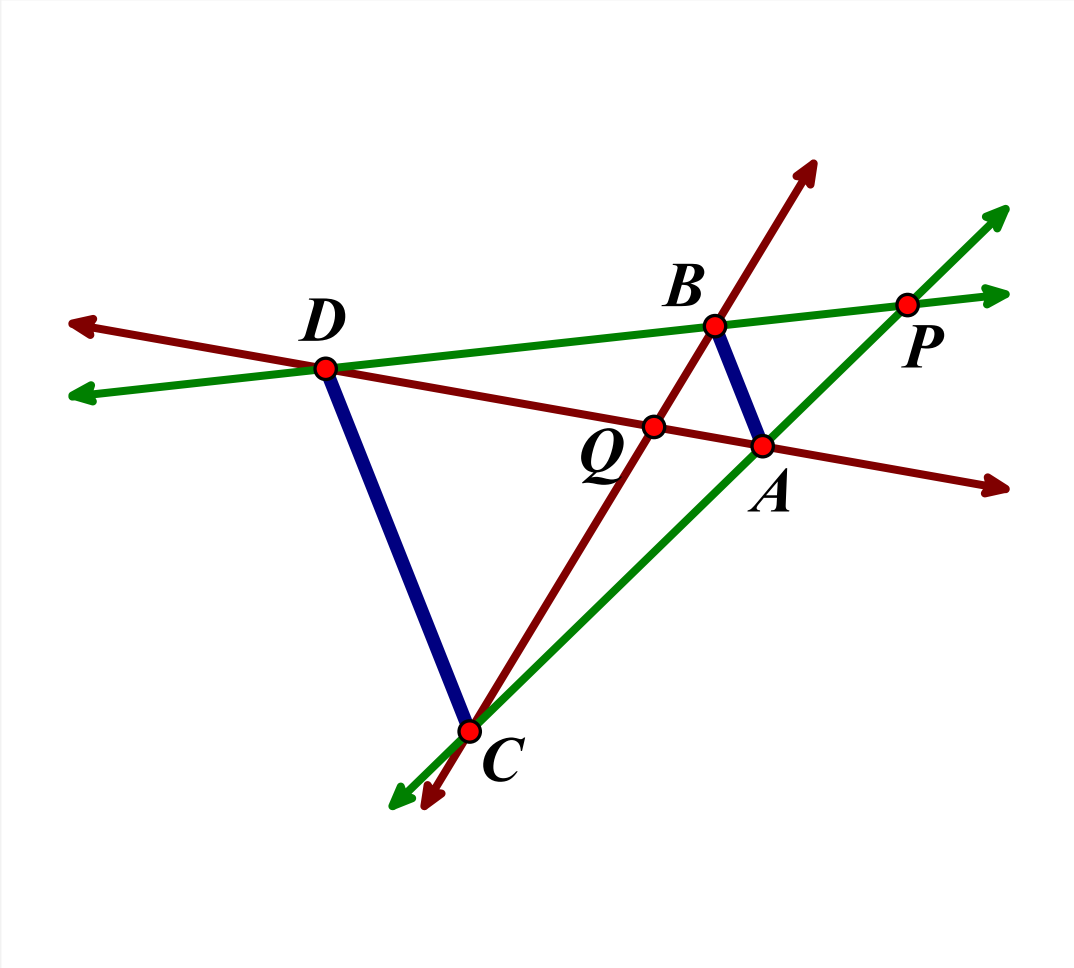
DILATION AXIOM IMPLIES THE EUCLIDEAN PARALLEL POSTULATE

- Given P and m.
- Construct PM perpendicular to m. Let line q be any lint **not** perpendicular to line PM.
- Let Q be on q and QN be perpendicular to PM.
- Dilate triangle PNQ with center P and ratio |PM|/|PN|
- Line n dilates to m, since both lines are perpendicular to PM and line q dilates to itself.
- So, Q', the image of Q, is on both m and q.



DILATION AND PARALLEL SEGMENTS

- Given two parallel segments AB and CD of different lengths.
- Lines AC and BD intersect at a point P. Lines AD and BC. intersect at a point Q.
- P and Q are centers of unique dilations, one takes AB the CD and the other takes AB to DC. One has positive ratio k = |CD|/|AB| and the other has **negative ratio** – k.. P and Q are **centers of similitude.**
- The sign indicates whether the two ordered segments have the same direction. Half-turns are dilations with ratio -1.

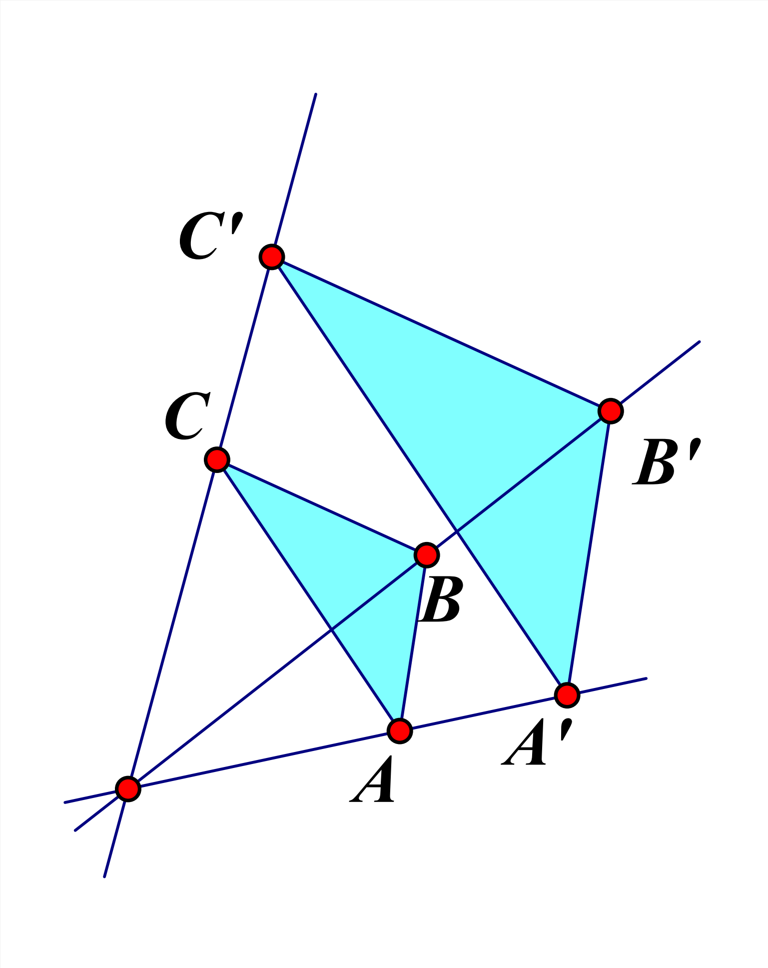


TRIANGLES WITH CORRESPONDING SIDES PARALLEL ARE SIMILAR

- Suppose that ABC and A'B'C' are two triangles with these pairs of sides parallel:
- AB and A'B', BC and B'C', and CA and C'A'

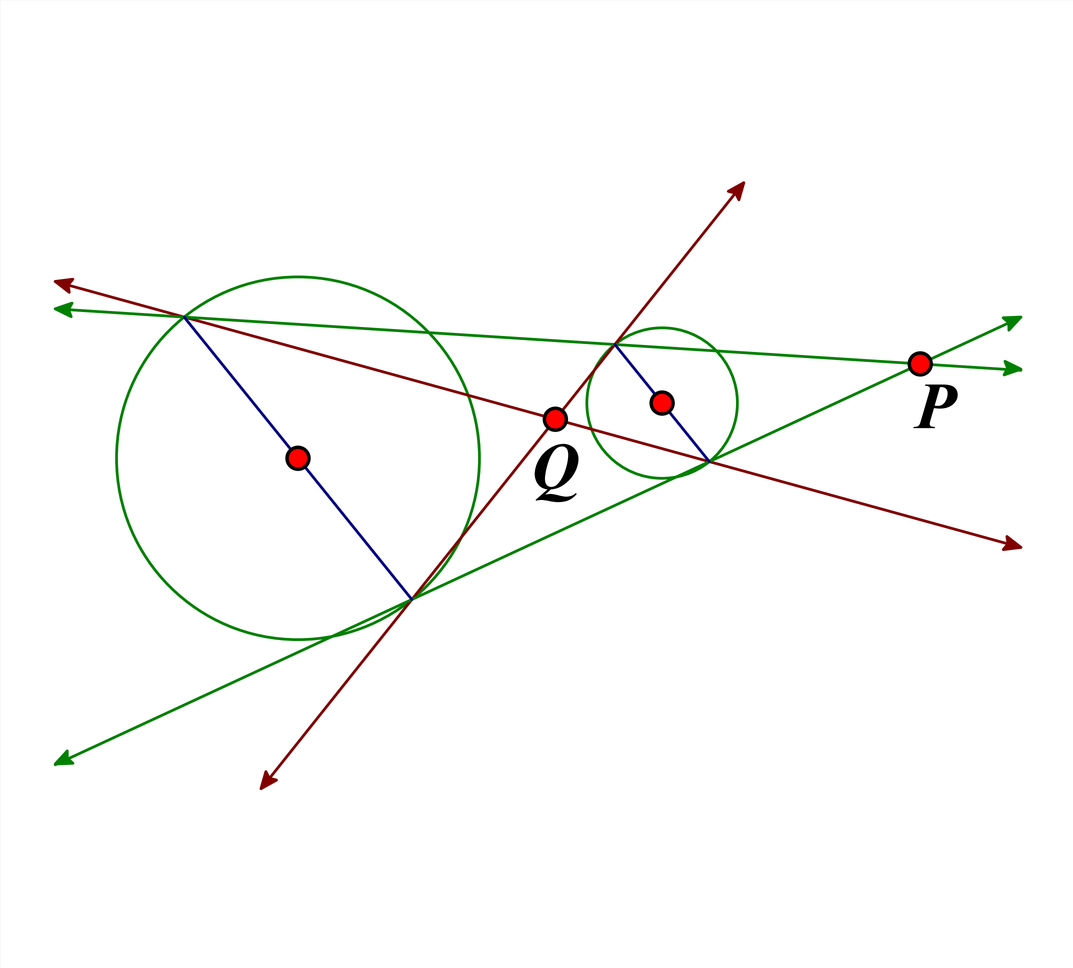
Then the two triangles are similar.

Proof; There is a unique dilation that takes AB to A'B'. This dilation will also take the entire triangle ABC to A'B'C' since the image of a line is a parallel line. So, the image of line BC is line B'C' and the image of line CA is line C'A'.



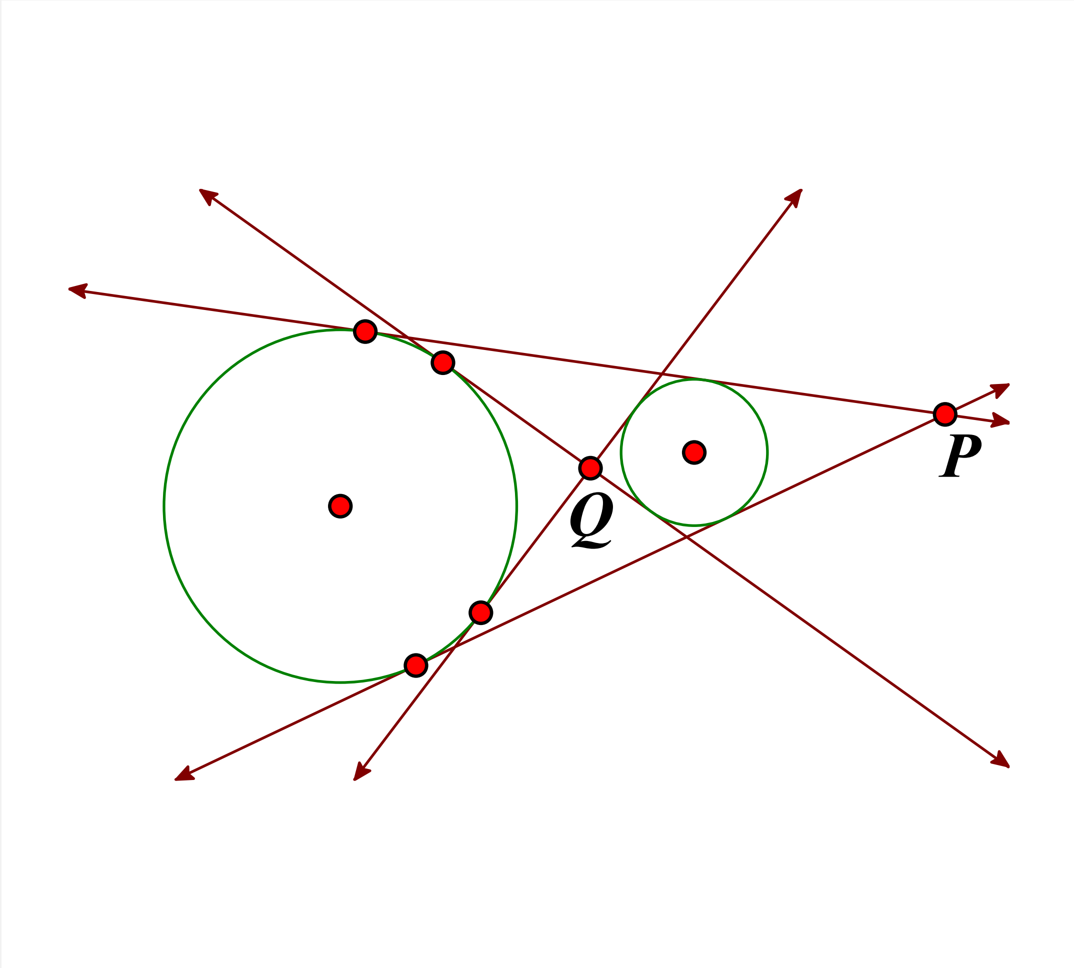
CIRCLES AND CENTERS OF SIMILITUDE

- Given two circles of different radius, the centers of similitude for any pair of parallel diameters are centers of similitude for the two circles.



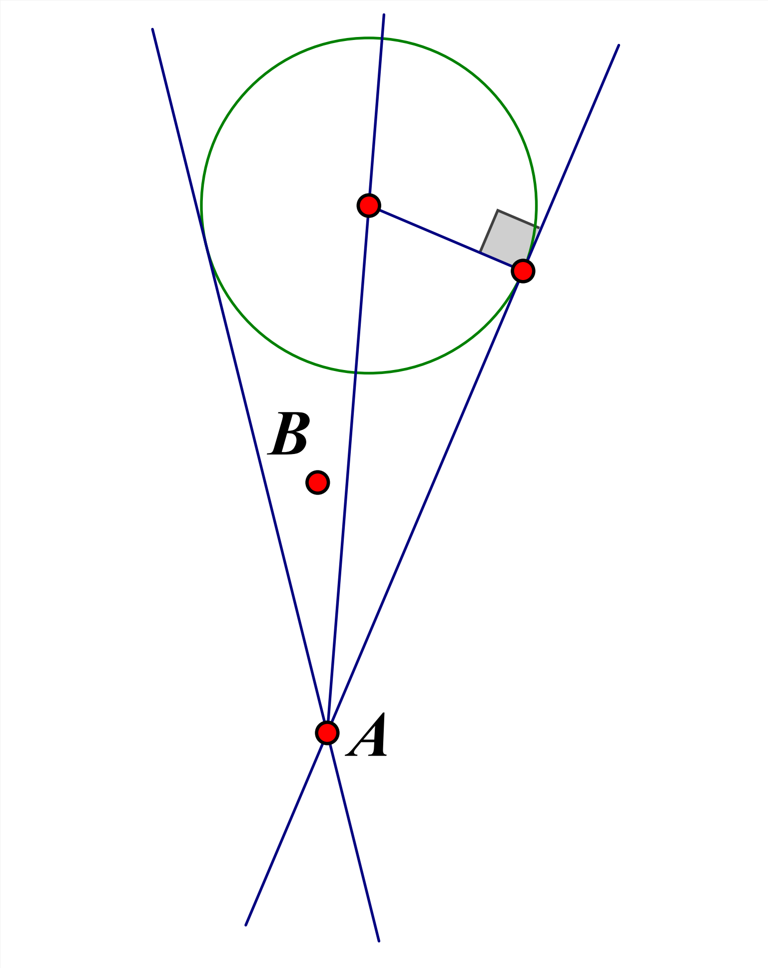
COMMON TANGENTS AND CENTERS OF SIMILITUDE

- The centers of similitude P and Q of two circles may be within both or outside of both circles,
- If such a center is outside both, any tangent line to one circle through the point is also tangent to the other circle.



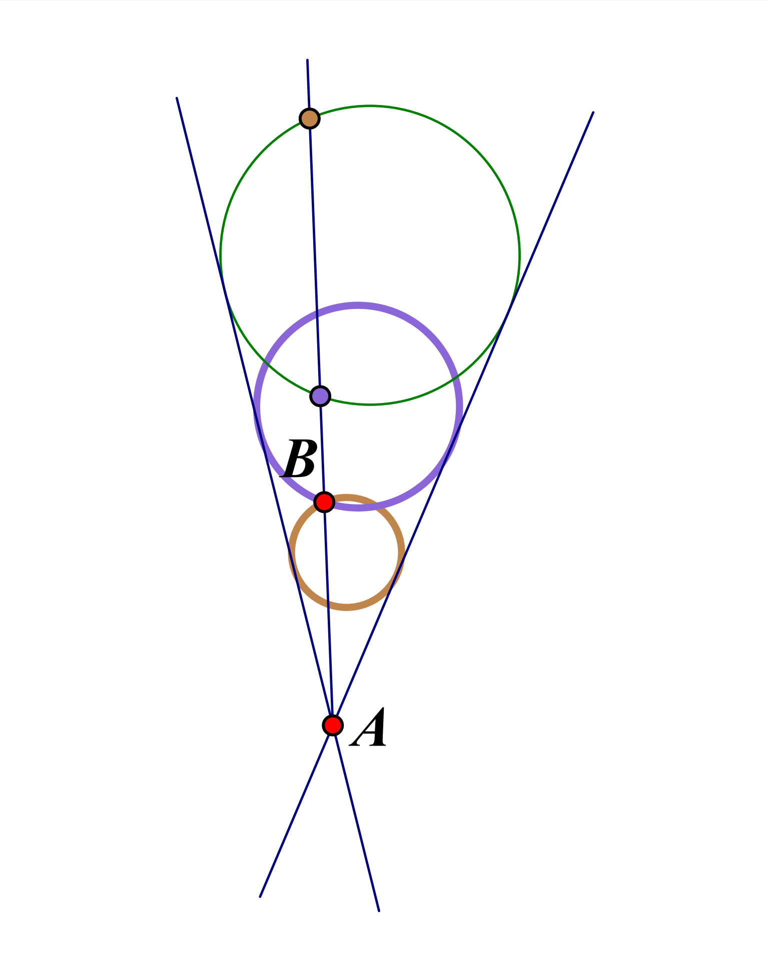
A CIRCLE CONSTRUCTION PROBLEM

- Given two lines and a point B not on either line, as in the figure, construct a circle tangent to both lines passing through the point.
- It is easy to construct a circle tangent to both lines by choosing a point on an angle bisector as center, then dropping a perpendicular segment to be a radius.



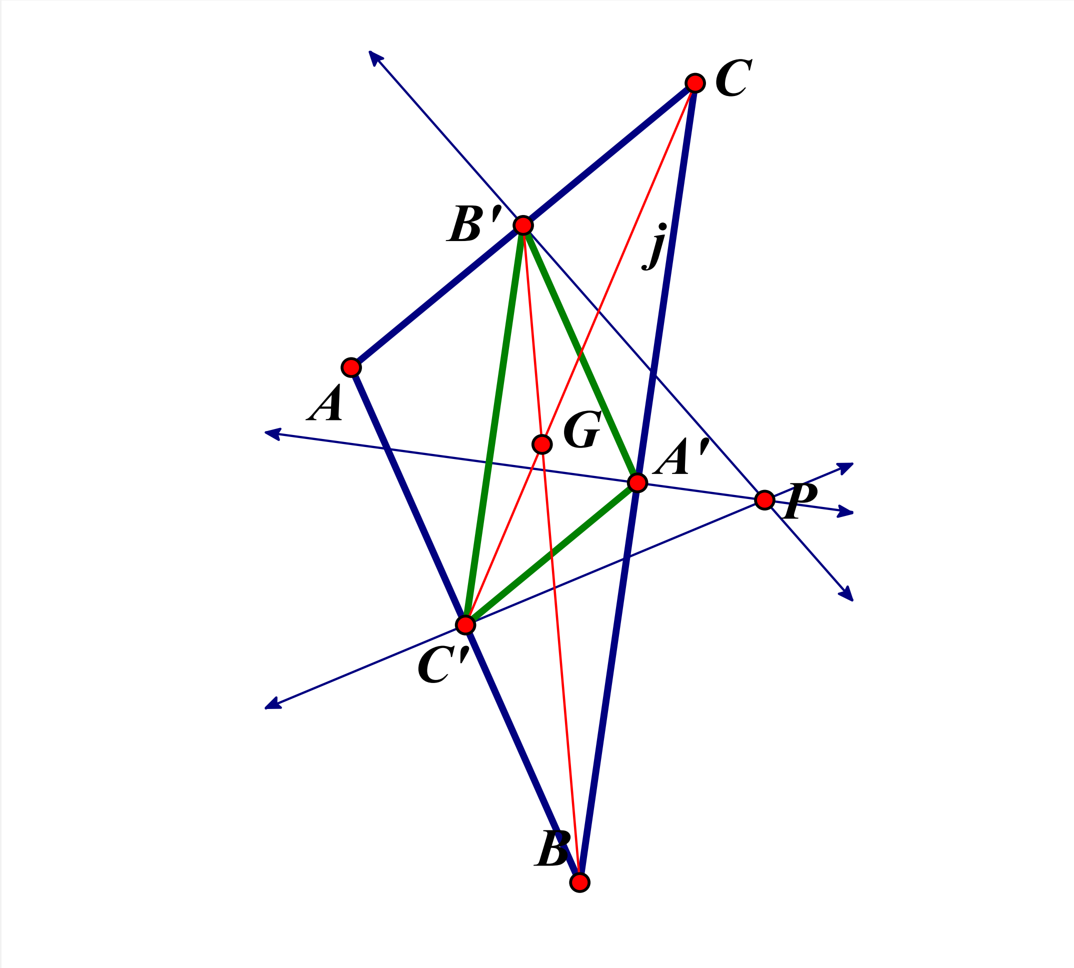
A CIRCLE CONSTRUCTION SOLVED BY DILATION

- A solution to the problem can be obtained by dilating with center A the circle tangent to the two lines.
- One can construct two points on the circle that can be dilated to B by intersection the circle with line AB.
- A dilation with center A that sends one of these points to B will dilate the given circle to a solution of the problem.



SOLVING CONCURRENCE OF ALTITUDES BY DILATION

- Given a triangle ABC, the perpendicular bisectors of the sides are concurrent at a point P equidistant from A, B, and C.
- For the midpoint triangle A'B'C', these p.b. of ABC are concurrent altitudes of A'B'C'.
- But there is a dilation of ratio -2 with center the intersection G of BB' and CC' that maps the altitudes of A'B'C' to the altitudes of ABC, which must also be concurrent. G is CENTROID ABC!



TEACHING SUGGESTION: RATIOS AND SIMILARITY

- If triangles ABC and DEF are similar with scaling factor k, then the **internal ratios** such as AB/AC and DE/DF are equal, but the ratio will not be the ratio of k = |DE|/|AB| of the similitude.
- Students may be confused by the two kinds of ratios.
- If seems to help to refer to the scaling number as the scaling FACTOR and avoid word ratio for this number, at least at the outset.
- It seems to help that the scaling factor is associated with a transformation, so can be visualized as a kind of motion.

SOME WAYS TO INTRODUCE RIGID MOTIONS INTO A GEOMETRY COURSE

- From the more formal to the more informal.

1. Replace the SAS axiom in your current approach with the Reflection Axiom. As soon as possible, prove SAS with this new set of axioms as we did here.
2. Instead of replacing the SAS axiom, just add the Reflection Axiom. Now you have more axioms than you need, but you can reason with them all the same.
3. Although all the other rigid motions are reflection products, in a less formal course, just assume they all exist from the start.

ADVICE: RELAX, MAKE SMALL CHANGES

- When talking about Rigid Motions as a kind of revolutionary idea, let's remember that in the end we are talking about the same Euclidean Plane as before, with a few extra tools. The same stuff is true. Once you get through the introduction, all the same proofs will still work.
- There is NO reason to include transformations in every proof, The best style is to use the right tool for each job, You can use several approaches and discuss the differences with your students.
- Don't be overambitious in making changes. It is better to make some modest changes and see how they work out, especially if you are happy with how you already are teaching geometry..

LOOKING FORWARD TO LATER MATH

The introduction of rigid motions and similarities, fits well into many future topics.

- Orientation is preserved by even products of line reflections. How about oriented congruence?
- Lots to do with symmetry and even self-similarity and fractals.
- The definitions of rigid motion and congruence extend to 3-space and STEM applications.
- In the (x,y) plane, transformations can simplify some formulas. Plane transformations are lead-in to matrix transformations in n-space.

RESOURCES FOR LESSONS

- My presentations at this page at UW: : **www.math.washingon.edu/~king/write**
- H. H. Wu's homepage at UC Berkeley Math Department. Wu was really a motivator for Common Core geometry. This page has links to two of his books that offer examples for teaching with rigid motions.
- Any treatment of isometries or symmetry in a geometry book, the Math Teacher, or online source.
- Online sources for the Common Core, or new textbooks such as the ones from Illustrative Mathematics. The library of geometry explorations for Geogebra.
- Maybe some ideas from my book, *Geometry Transformed*.

EYES ON THE PRIZE

- The goal is for your students to have a rich understanding of geometrical relationships and how to reason about them.
- There are many paths to get there.
- And there should be some beauty and enjoyment along the way.