# Reasoning and Exploring with Rigid Motions In Geometry by Jim King

- Welcome to this minicourse.  What we will be doing this morning is based on an online course created for the IAS/Park City Mathematics Institute.  I have modified some slides, but I have left the PCMI logo in the corner.

**When you arrive, please fill out the short form on the table.** This info will help me adapt our time to serve who is here.

# Agenda for Part 1

1. Course Goals
2. Congruence: Concept and Definition
3. Rigid Motions and Line Reflections
4. Congruence in isosceles triangles
5. Reflection Axiom
6. Congruence of Triangles
7. Some Standards from the Common Core

# Course Goals

- Insight into the "big picture" of the value added by basing early geometry concepts on rigid motion transformations
- Connections between formal reasoning and real-world visual and physical informal reasoning
- Hands-on, detailed exploration of rigid motions and their compositions, with a bit about dilations and similarity
- Practice in proving propositions and problem-solving using transformations
- Consideration of the role of coordinates

# Definitions of Congruence

Before class, we asked you: What is the definition of congruence that you use in your class, or what definition do you see in textbooks?

Also, in your shape sorting, what method(s) did you use to sort?

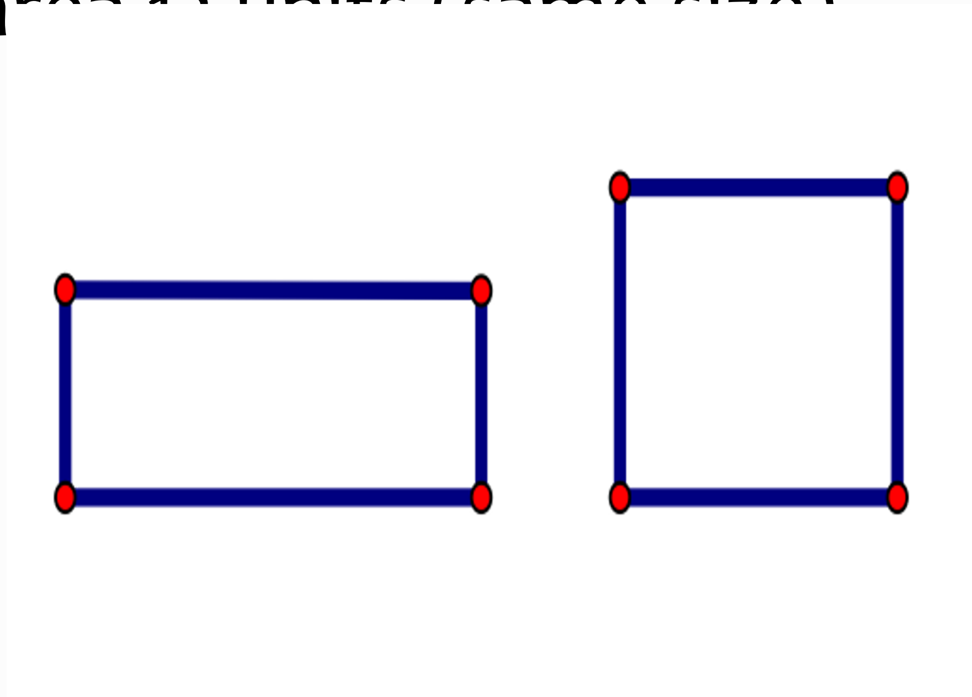
# Some examples to ponder

- Here are some examples that illustrate problems with the language or limitations of many textbook definitions.

F is congruent to G
if …

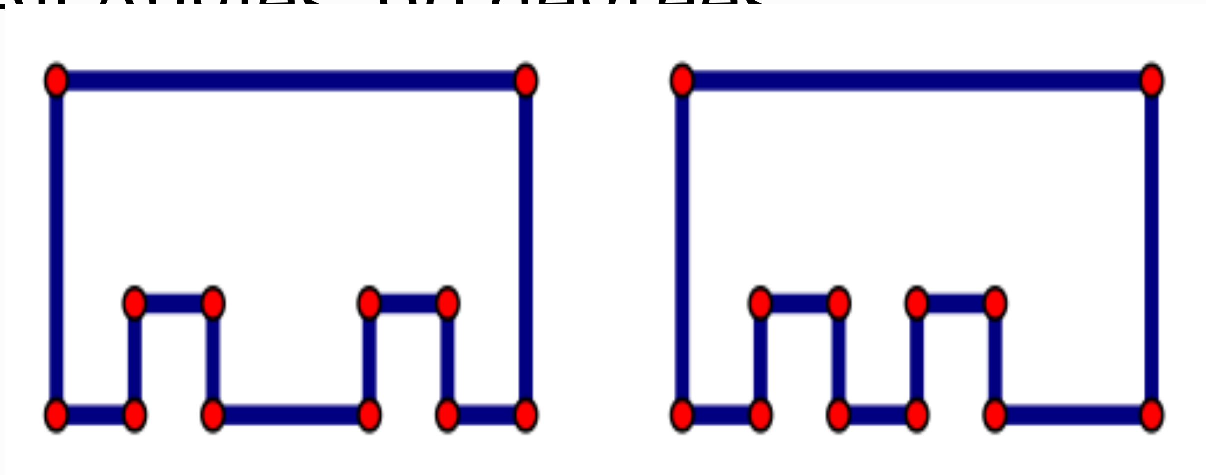
# “Same Size and Same Shape”

Both rectangles (same shape) with area 12 units (same size)



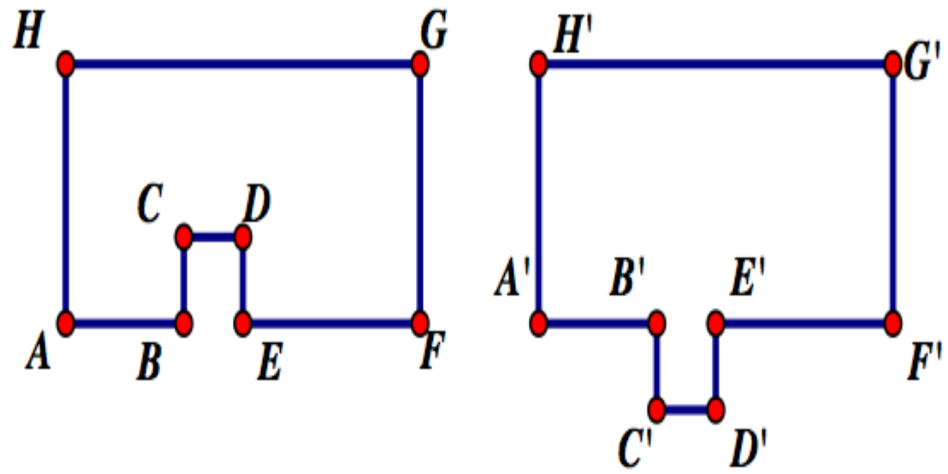
# “Equal lengths and Equal Angles”

- Side lengths in each: 6, 3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1
- All Angles: 90 degrees



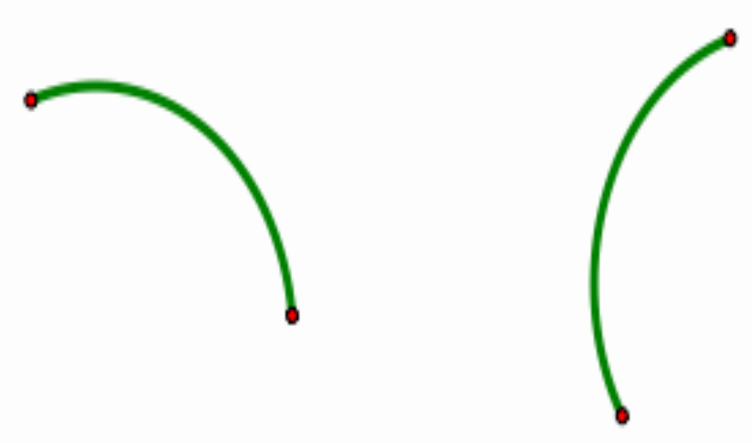
# “Corresponding Lengths, Angles Equal”

- Corresponding side lengths equal, all angles are right angles.



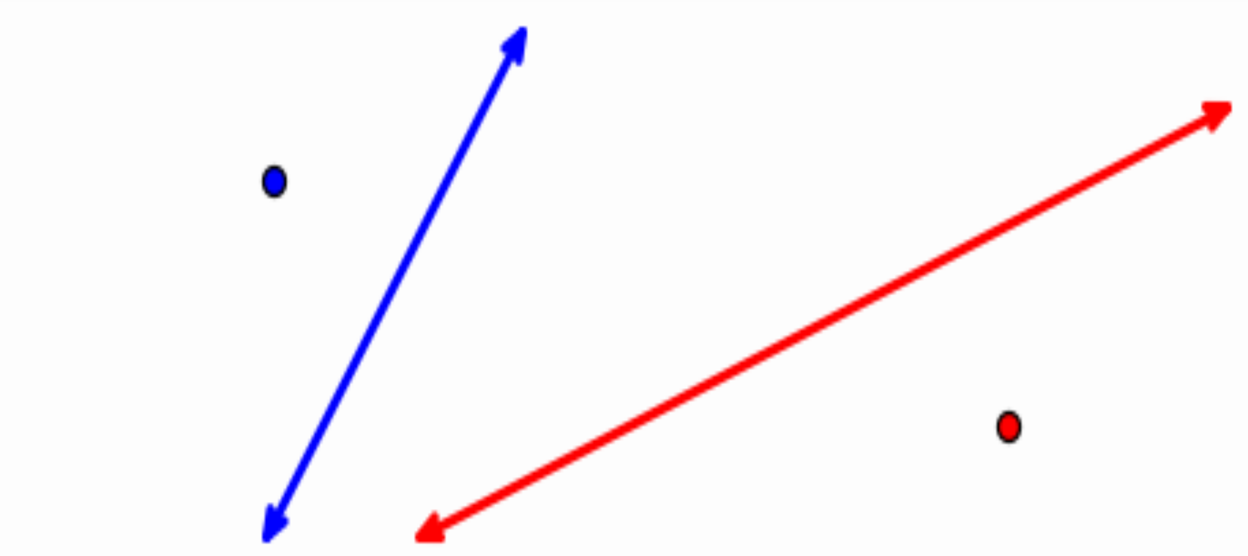
# Non-Polygonal Shapes

- Are these arcs congruent? What are the corresponding lengths and angles?



# Figures that are not connected

- Each figure consists of one point and one line. Are they congruent? Again, what



# Intuitive Concept of Congruence

We may have a ***concept*** of congruence that is expressed in words like this:

- "**Move** one shape to that it matches the other."
- "**Cut out** one shape and lay it exactly onto the other."
- "**Superimpose** one shape onto the other."

# Superposition

This is heading down the right track, but we need a way to express “move” and “cut out” and “superimpose” in language we can use in mathematical reasoning.

The Greeks did not have a word for it, but after 2000 years, we do. The word is ***function***, or – in this case – ***transformation***.

# Definition of Congruence

We say a set A is *congruent* to a set B (and write $A \cong B$) if B is the image of A by a rigid motion.

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To do list:

- Define *rigid motion.*
- Show that this relationship is *symmetric*.….even more, an *equivalence relation*

# Rigid Motions

- Definition:  A ***rigid motion*** of the plane is a transformation of the plane that preserves distance measure and angle measure.

- This is our candidate for a ***mathematical model*** of the physical act of picking up a sheet of paper from a table and setting it down again in a new position.

# Preserving Distance and Angle Measure

- A transformation T ***preserves distance*** if for every pair of points A and B, the distance
$$|AB| = |T(A)T(B)|,$$
the distance between the image points.

- A transformation T ***preserves angle measure*** if for any angle $\angle BAC$, these angle measures are equal:
$$m\angle BAC = m\angle T(B)T(A)T(C)$$

# A is congruent to B: How Different?

| Textbook: | CCSS: |
|---|---|
| Corresponding lengths of A and B are equal and corresponding angles are equal | B is the image of A by a rigid motion. |

Correspondence

Equal Lengths

Equal Angles

Transformation

Preserves Distance

Preserves Angle Measure

# Three Transformations

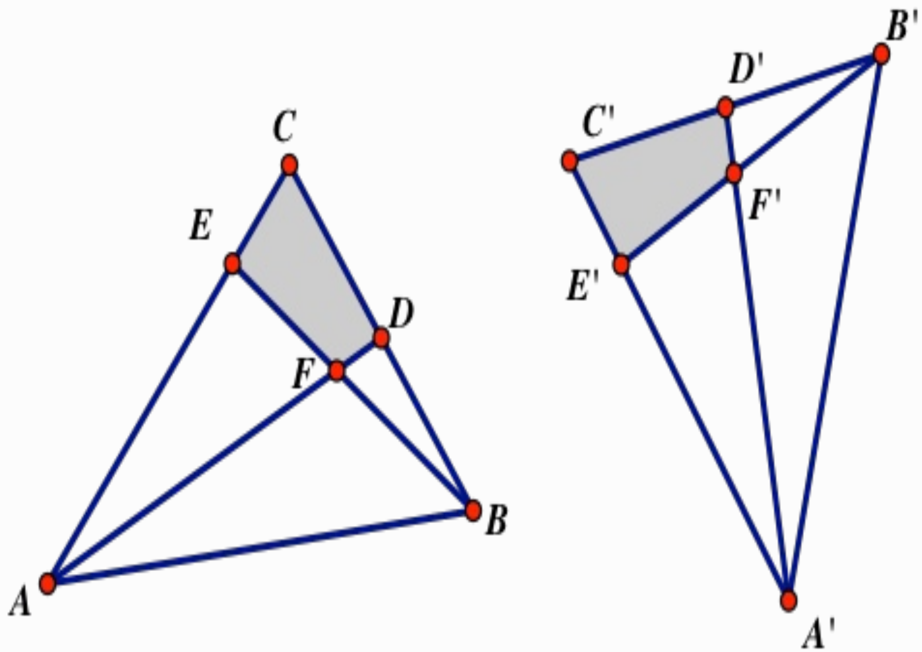
- The Common Core names three kinds of transformations that turn out to be rigid motions.
  - Reflections
  - Rotations
  - Translations
- In fact, the Common Core geometry path ***assumes as a postulate*** that these three are rigid motions. (More about this later.)

# Advantages of this Congruence Definition (A Sales Pitch)

- It is a real mathematical definition based on defined terms. It can be checked when examples arise.
- It applies to all kinds of sets in the plane, finite or infinite, bounded or unbounded, connected or not.
- It corresponds to our intuitive idea of superposition.
- It brings along all the points of the plane that may show up in your reasoning (see the next slide).

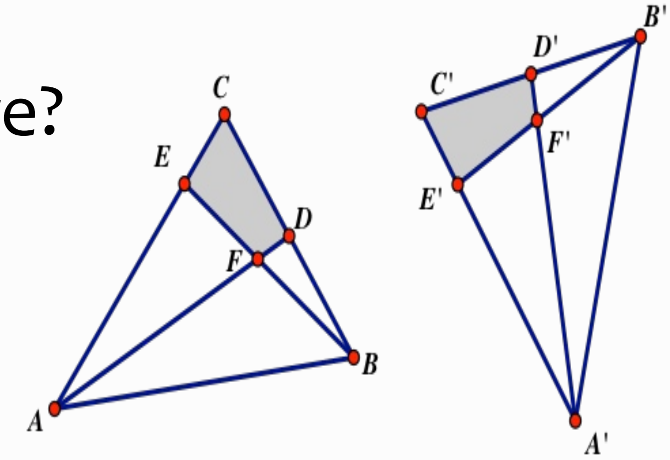
# Example of a Congruence Problem

Given two triangles, ABC congruent to A'B'C'.  Let D be the midpoint of BC and D' be the midpoint of B'C'.  Also, d(C,E) = d(C',E'). Prove that the quadrilateral DCEF is congruent to D'C'E'F'



# Real World and these Figures

- The figures in the problem were part of your hands-on sort.
- Do you feel sure that the quadrilaterals in your hands are congruent?
- What makes you feel sure?



# Preview of a Solution using Rigid Motions

(Assume already proved that rigid motions map line segments to line segments.)

1. Since triangles ABC and A'B'C' are congruent, there is a rigid motion T with T(A) = A', T(B) = B', T(C) = C'.
2. Then T(D) = D' and T(E) = E' since distances are preserved.
3. So F, the intersection of segments AD and CE, is mapped to the intersection of segments A'D' and C'E', which is F'
4. By definition the quadrilaterals DCEF and D'C'E'F' are congruent, since T maps the vertices (and hence the edges) from one to the other.

# A Foundational Interlude: Basic Axioms

- We need a few basic properties (axioms) for plane geometry to get started.  Since these are familiar, we will try to skate over them quickly and hope this suffices.
- But if you have questions,
  (1) please ask and also
  (2) you have a handout with details.

# Breakout 1. Perpendicular Bisector

- Given distinct points A and B, on a piece of paper, use your ruler and protractor to draw (a finite part of) the perpendicular bisector of segment AB.
- Where do you use the ruler?  The protractor?
- Note: This works But you cannot "drop" a perpendicular to a line just using the basic axioms. There is no way to place the protractor center.

# Rigid Motions and Lines

Suppose T is a rigid motion.

1. **T maps lines to lines.** From the protractor axiom, P is on line AB when $m\angle BAP = 0$ or 180. The image angle $\angle T(B)T(A)T(P)$ will have the same 0 or 180 measure. also = 0 or 180.
2. **If P is on line AB, then the image T(P) is determined by T(A) and T(B).** From the ruler axiom, a point P on line AB, is completely determined once we know $|AP|$ and $|BP|$. So T(P) is determined since we know its distance from T(A) and T(B)
3. **If A and B are fixed points, all the points on line AB are also fixed points.** This is a special case of 2.

# Reflection Axiom

- This assumption is key to the geometry in the Common Core. It is the first big difference from most textbooks.
- **Reflection Axiom:** For every line m in the plane, there is a rigid motion, not the idenity, that fixes the points of m.

Key Point: The geometric recipe for line reflection will be a consequence of its being a rigid motion that fixes the points of the line.

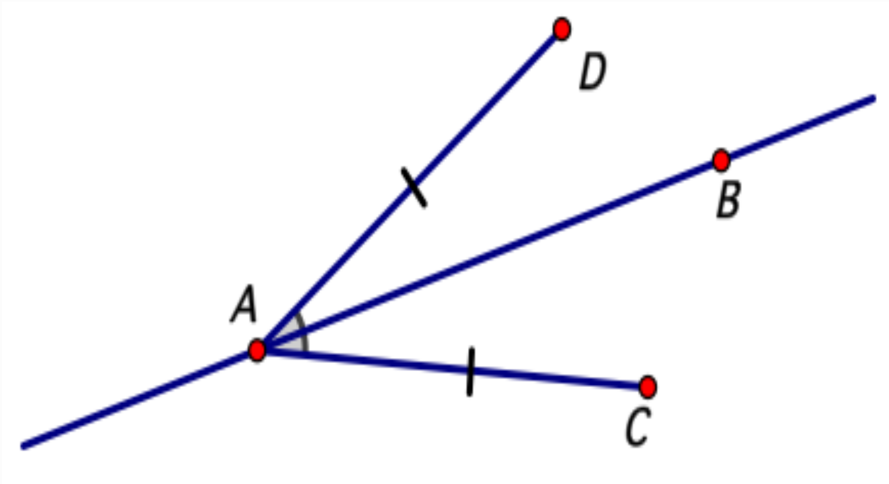
# Breakout 2: Fixed points, folds, mirrors

Work in pairs or threes on your breakout worksheet with tasks on

1. Modeling fixed points
2. Modeling with paper folding and "Miras".
3. Constructing the image T(C) if A and B are fixed.

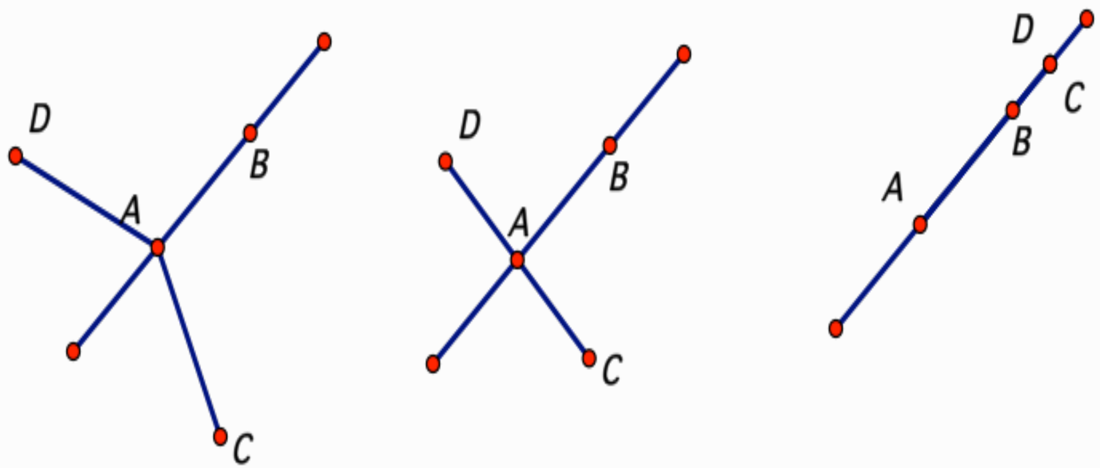
# Results of Breakout 2

- I hope that we arrived at a conclusion that if C is not a fixed point, then the image D of C must look like this figure, which we call the mirror figure, with the equal lengths and angles marked.



# Other Cases of the Mirror Figure

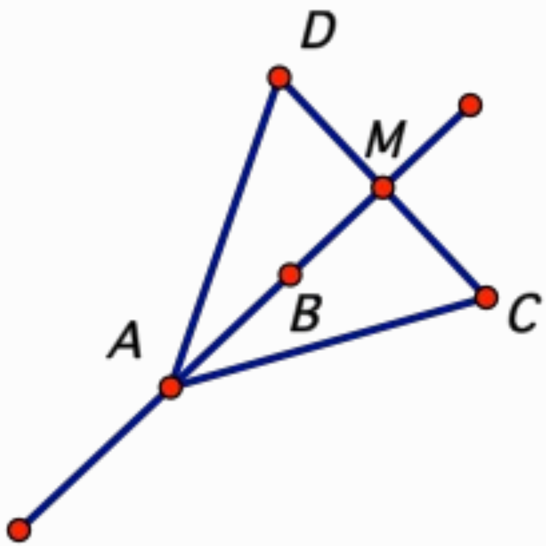
- The mirror figure may look variously depending on the location of C. If C is on the line, then D = C. If angle BAC is a right angle, AB is the perpendicular bisector of CD.



# Fixed Points and Perpendicular Bisector

A rigid motion T fixes A and B. In your mirror figure of C and the image D = T(C), draw segment CD.
Let M be its intersection with line AB.

- Angles AMC and AMD are congruent right angles. Why?
- AM and CM are congruent, so M is the midpoint of CD. Why?
- Therefore, line AB is the perpendicular bisector of CD.



# Three Fixed Points imply Identity

- Let T be a rigid motion that fixes three distinct points A, B, C. We claim that all other points P are also fixed.
- If P were a point not fixed by T, we have shown that each line AB, BC, CA is the perpendicular bisector of segment PT(P). Therefore, the lines are the same and A, B, C are collinear.
- Thus, if a rigid motion has three non-collinear fixed points, it must be the identity.

# Geometric Description of Line Reflection

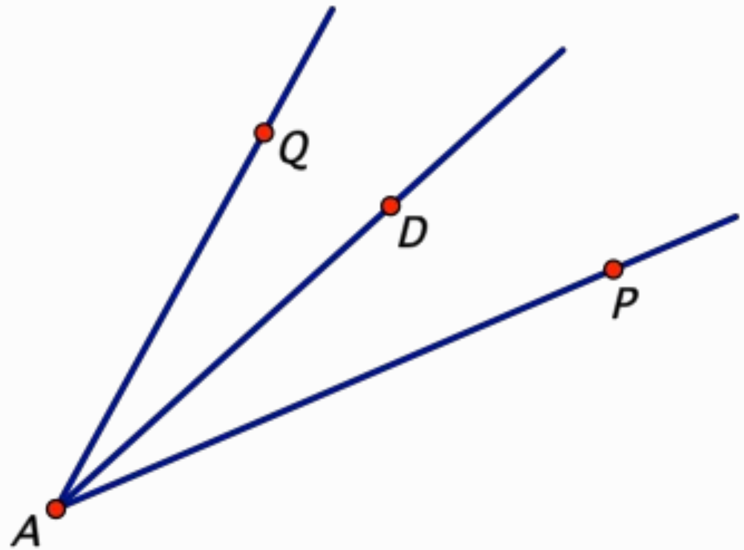
- Let T be a rigid motion which fixes the points of m but is not the identity. We have now shown there is exactly one such T.  We call it reflection in m and denote it by $R_m$.
- There is only one because, for C not on m, $R_m(C)$ is determined uniquely by the mirror figure
- $R_m(C)$ is the unique point so that m is the perpendicular bisector of $CR_m(C)$.

# Isosceles Relationships

- You may have seem some isosceles triangles in what we have already done. We are going to establish a bit of elementary triangle geometry and then use it to prove our fundamental congruence test for triangles.

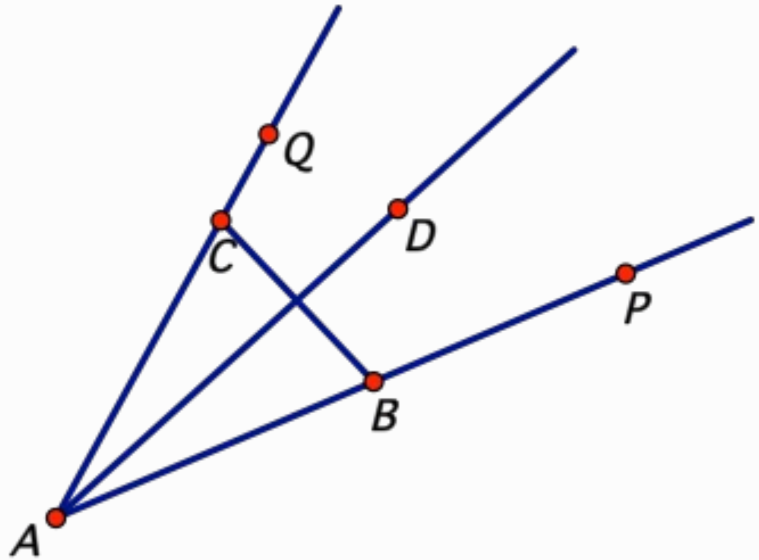
# Angle Bisector

- Given ∠PAQ, by the protractor axiom there is an angle bisector ray AD. In other words, AD is interior to ∠PAQ and m∠PAD = m∠DAQ.
- Let t = line AD. Reflection $R_t$ maps ray AP to ray AQ and the reverse



# Angle Bisector of an Isosceles Triangle

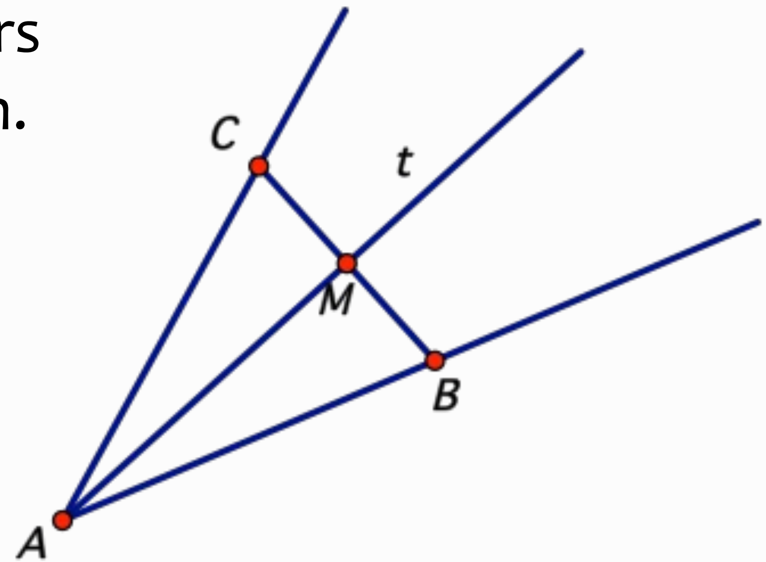
- Let t = line AD as before. If points B and C are on rays AP and AQ, with |AB|=|AQ|, then triangle ABC, with two sides of equal length, is called an isosceles triangle.
- Then $R_t(A) = A$, $R_t(B) = C$, $R_t(C) = B$.



# Breakout 3: Isosceles Analysis

Triangle ABC is an isosceles triangle. Line t bisects $\angle$BAC. M is the intersection of *t* and BC.

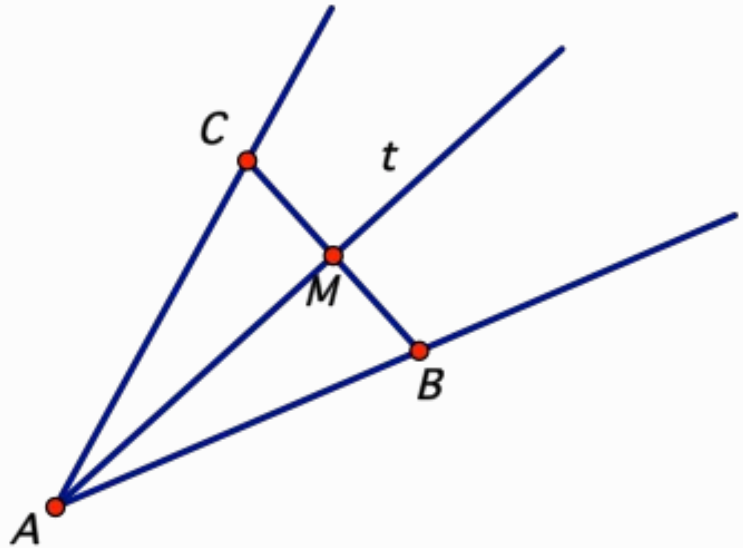
- Work in twos or threes. Write down as many examples of pairs of congruent figures as you can.
- How do you know they are congruent?
- How many can you find?



# Congruent Objects in an Isosceles Triangle

- Here is a list of congruent objects in the figure. Did your group list them all? Are there more? Are the reasons clear?

$AB \cong AC$
$MB \cong MC$
$\angle MAB \cong \angle MAC$
$\angle ABC \cong \angle ACB$
$\angle AMB \cong \angle AMC$
$\Delta AMB \cong \Delta AMC$
$\Delta ABC \cong \Delta ACB$



# What about Rotations and Translations?

- We will be using and studying rotations and reflections soon.
- But for this course we decided to focus on reflections for our first steps because
  - We want a single, simple way to get to big congruence theorems.
  - Reflections may be less familiar, but they are easier to define.
  - We will see that in some sense reflections are the basic building blocks for all rigid motions, so we will not need an Axiom for Rotations or Translations.

# Before Proving Side-Angle-Side

- Our big goal for this section is to prove the Side-Angle-Side (SAS) criterion for congruence of triangles.
- But before we get that far, there are important first steps. The first is to prove this

**Any two points are congruent.**

Think a minute. Why does this need a proof?

# Any Two Points are Congruent

- Of course this must be true or there is something wrong with our model. But our definition is not about size or shape but about finding a rigid motion.
- So given distinct points A and B, let m be the perpendicular bisector of AB. We have seen that $R_m(A) = B$. So A is congruent to B.

# Next step before SAS

- Our big goal is still to prove the Side-Angle-Side (SAS) criterion for congruence of triangles.
- The next step towards this goal is to prove this:

**Two segments of equal length are congruent.**

# Congruent Segments

Proposition: Given two segments AB and CD such that $|AB| = |CD|$, there is a rigid motion T with $T(A) = C$ and $T(B) = D$. Therefore, $AB \cong CD$.

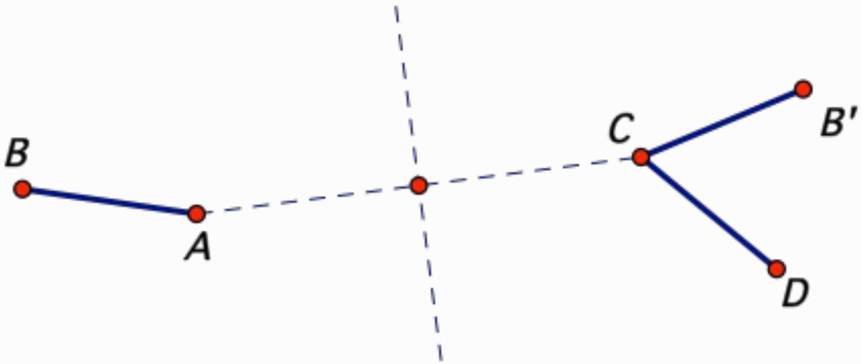
Special note: We need T to prove congruence, but we will see that the existence of T is a bonus that we will use in many proofs.

# A Rigid Motion: AB to CD (part 1)

Assuming |AB| = |CD|, let m be the perpendicular bisector m of AC, then the reflection of A in m will be C. Therefore, the $R_m$-image of AB is a segment CB’, with |CB’| = |CD|.

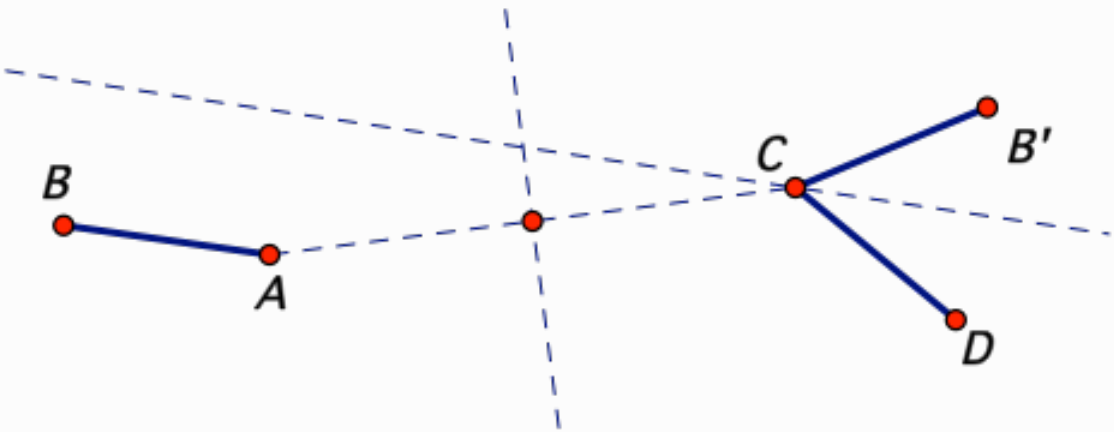
1. If CB’ coincides with CD, that is if B’ = D, then T = $R_m$ does the job, and our proof is complete.
2. If not, then we have a figure something like this We notice that DCB’ is the isosceles figure we investigated in our breakout session.

Notice that the congruence of A and B was essential here!



# A Rigid Motion: AB to CD (part 2)

- If n is the line bisecting angle DCB’, then $R_n(C)=C$ and the image of B’ is D. So the image of CB’ is CD.
- This shows AB is congruent to CD, with $T = R_n\, R_m$.



# So … Congruence of Segments

- If the two segments have the same length, why did we have to prove congruence?
  - While we want segments of equal length to be congruent, we did not take this as the definition of congruence, so a proof was needed.
  - To some extent, this was a test to see how well our definition matches our conception of congruence.
- Will we have to do this in every proof?
  - NO NO NO! This can be a once-in-a-lifetime experience. You can use this without proving it ever again (unless you are teaching it!).
- We have key information that we can use without re-doing the proof!
  - Given two segments of the same length, one can be mapped to the other by a reflection or a composition of two reflections.

# Connecting with the Common Core

- The choice to focus first on reflections for this short course for teachers is not necessarily what a teacher would do with students.
- Let’s look at a bit of the Middle Grades Common Core Math and see how it relates to this module.

# Common Core: 8th Grade

1. Verify, experimentally the properties of rotations, reflections, and translations:
   a. Lines are taken to lines, and line segments to line segments of the same length.
   b. Angles are taken to angles of the same measure.
   c. Parallel lines are taken to parallel lines.

2. Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations: given two congruent figures, describe a sequence that exhibits the congruence between then.

# Worry, Worry?

- "OK, our definition of congruence is mathematically sound, but it seems abstract and awkward. Is this going to make geometry harder for me and my students?"
- -- *Not so!* You will see that in fact that this precision will bring us all the old geometry tools we love, but lots of cool new ways of thinking.
- Also, students can focus on the three named types of rigid motions rather than the more abstract general definition.

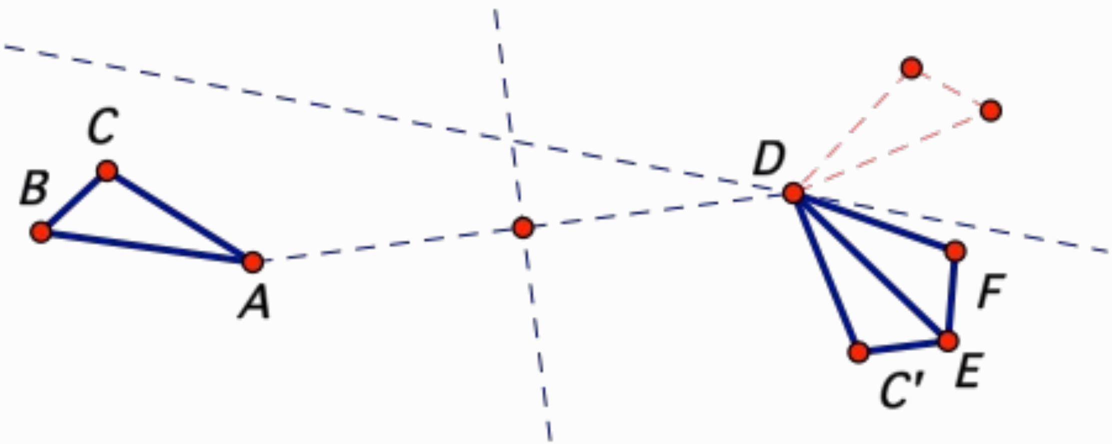
# Proof of SAS for triangles

Theorem (Side-Angle-Side): For two triangles ABC and DEF, if $|AB| = |DE|$, $|AC| = |DF|$ and $m\angle BAC = m\angle EDF$, then triangle ABC is congruent to triangle DEF.

- This is often abbreviated as SAS. SAS is traditionally assumed as an axiom. But instead, we have assumed the Reflection Axiom, that for each line, there is a rigid motion called reflection. Using this, we can prove SAS.
- In fact, we have already done almost all the work!

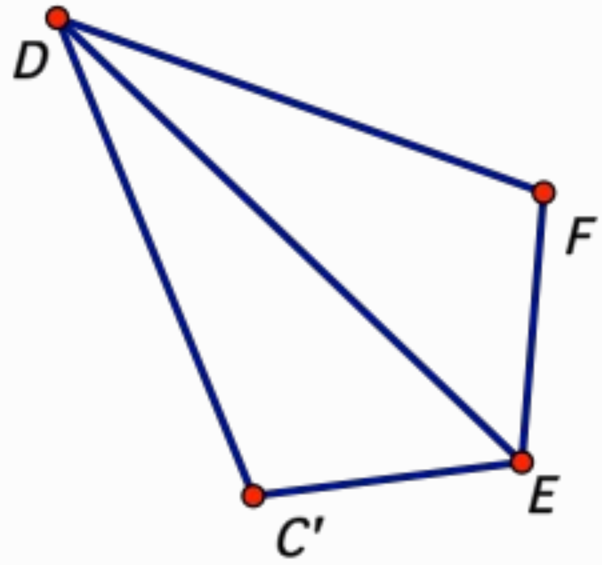
# Proof of SAS: Part 1

- We know from segment congruence that there is a rigid motion T that maps A to D and B to E. Then C' = T(C) is some point with $m\angle EDC' = m\angle EDF$ and $|C'D| = |FD|$.



# Proof of SAS: Part 2

- Since $m\angle EDC' = m\angle EDF$ and $|C'D| = |FD|$, C' may be on ray DF, with C' = F. In this case T defines the needed congruence.
- Otherwise, triangle FDC' is an isosceles triangle, and C' is $R_{DE}(F)$. In this case $R_{DE}T$ is a rigid motion that maps triangle ABC to DEF. Thus SAS is proved.



# SAS Loose Ends

- In the proof of SAS, we constructed a T that maps A to D, B to E and C to F.  T was the either a line reflection or else the composition (product) of two or three line reflections.
- One can use SAS to prove that two angles are congruent when they have the same measure.

# Congruence of Rectangles

- Let ABCD and KLMN be rectangles. If |AB| = |KL| and |AD| = |KN|, we can apply SAS to get a rigid motion T that maps triangle ABD to KLN. The images of points A, B, D are K, L, N.
- The T-image of line BC is a line through L, the image of B, perpendicular to LM (by angle preservation). This is line LM.
- Likewise the image of AD is line NM.
- Since C is the intersection of lines BC and DC, the image T(C) must be the intersection of lines LM and NM, which is M.
- So the T-image of ABCD is KLMN. This proves ABCD ≅ KLMN.
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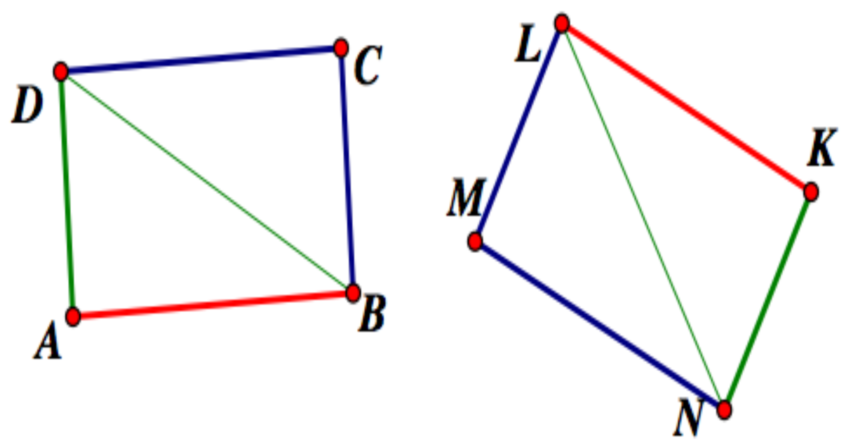

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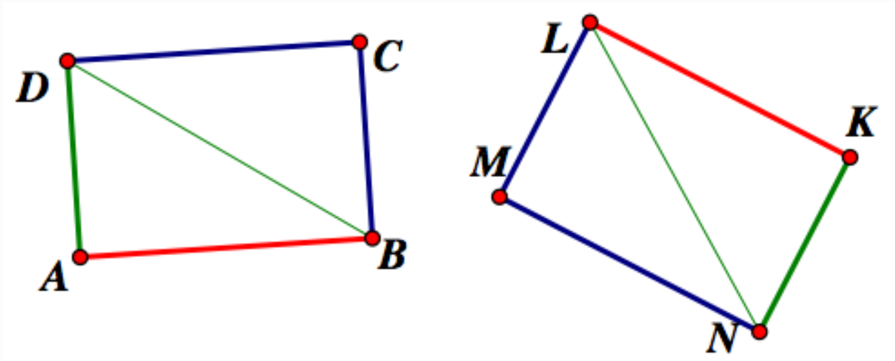
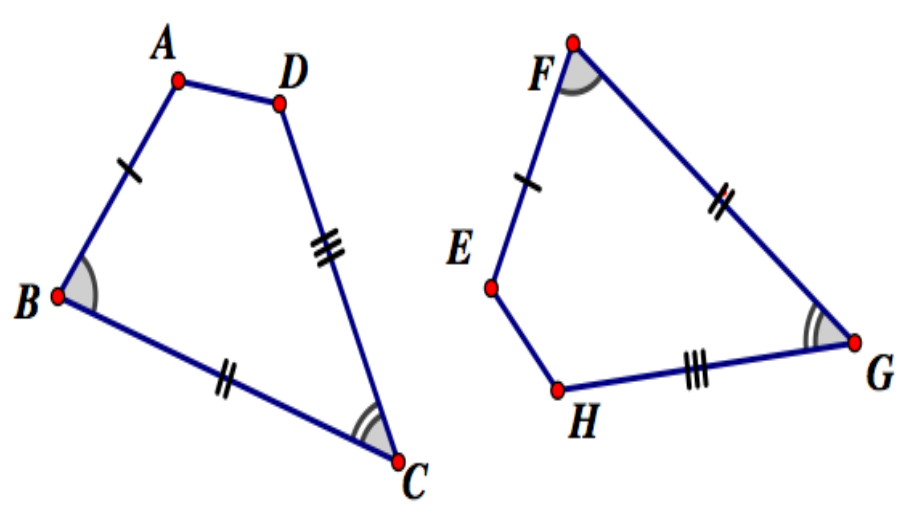


# Example: SASAS

- Let ABCD and EFGH be convex quadrilaterals, with
- $|AB| = |EF|$, $|BC| = |FG|$, $|CD| = |GH|$ and $\angle ABC = \angle EFG$, $\angle BCD = \angle FGH$
- Prove that ABCD is congruent to EFGH.

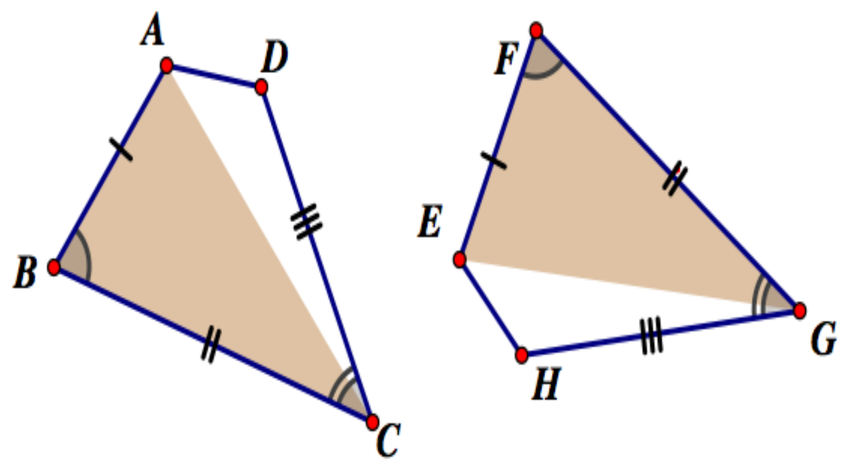
- The corresponding pairs above are marked in the figure.



# One Proof of SASAS

- A "traditional" proof of SASAS would break up each figure into two triangles and show by SAS that the corresponding triangles are congruent.
- This will prove that all 4 corresponding sides and angles are equal, but it will not easily produce a rigid motion to prove congruence.

Instead find a T!



# SSS and ASA

- There are two other important tests for triangle congruence.
- Side-Side-Side (SSS) and Angle-Side-Angle (ASA) can be proved along the same lines as SAS. Start with a congruence T that takes one side of the first triangle to a side of the second. Then either one is finished or else one now has two triangles sharing a side. One more reflection will finish the job.