Group 3:
Clint Chan
Karen Duncan
Jake Jarvis
Matt Johnson

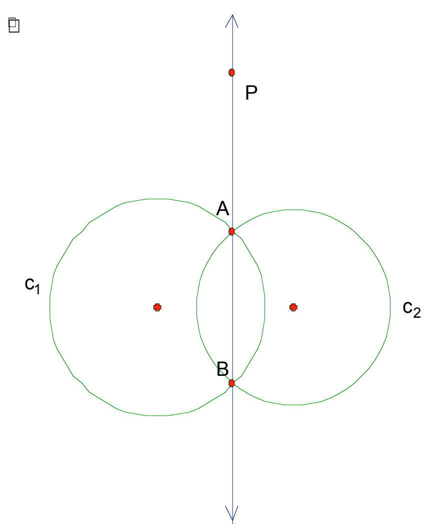
“The power of a point P with respect to a circle is the product of the two distances PA times PB from the point to the circle measured along a random secant” (B&B 261). When the point P is on the circle, its power is zero; when it is inside the circle, its power is negative; and when it is outside the circle, its power is positive. The power of a point when P is outside the circle is also equal to (PT)^2, where T is the point where the line through P is tangent to the circle. Also of interest, if P is outside the circle and e is a circle which is orthogonal to c and centered at P, then pc(A) = t^2, where t is the radius of e (from "The Power of a Point and Radical Axis" class notes).

The radical axis of two circles is, by definition, the locus of points A for which the power of A with respect to c[1] and c[2] are equal.
Using some facts about the power of a point, we can see that the radical axis is a line.

Fact 1: If A is a point outside the circle and S is a secant intersecting c at M and N, then the power of A with respect to c is equal to _AM__AN_.

Fact 2: If A is a point outside the circle and AT s a line tangent to c at T, then the power of A with respect to c is equal to _AT_^2.

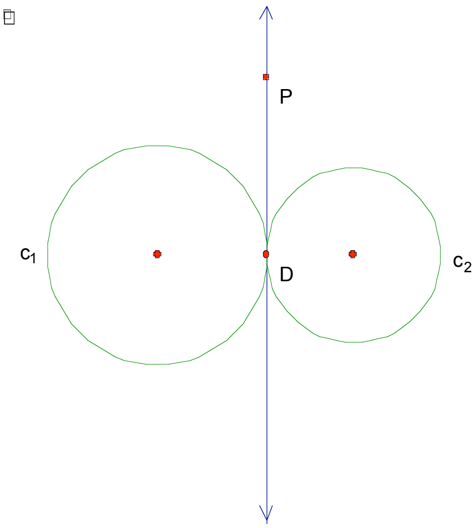
The easiest case to see this is when two circles c[1] and c[2] intersect in two places.



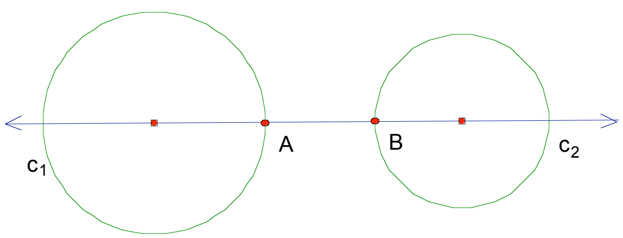
By fact 1, the power of any random point P on the line through A and B with respect to both c[1] and c[2] is equal to _PA__PB_. For any P not on this line, the power of p with respect to each of the circles is not equal.

If c[1] and c[2] intersect in just one point, we look at the shared tangent.

By fact 2, the power of any random point P on this tangent line with respect to both c[1] and c[2] is equal to _PD_^2. For any P not on this tangent line, the power of P with respect to each of the circles is not equal.



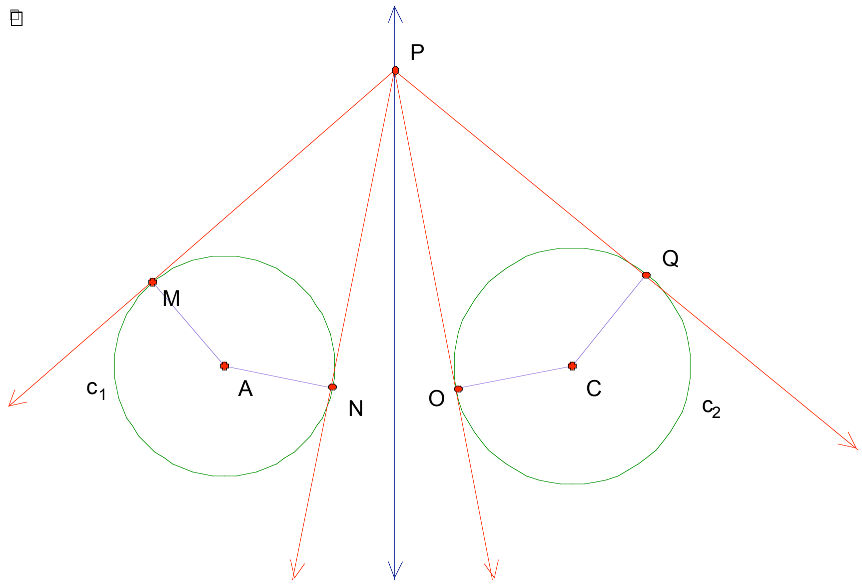
If c[1] and c[2] do not intersect, we look at the line connecting the centers of the circles.



Point A has zero power with respect to c[1], but it has some power with respect to c[2]. If we look at some point along line AB closer to c[2], its power with respect to c[1] will increase, and it's power with respect to c[2] will decrease. Point B has some power with respect to c[1], and zero power with respect to c[2]. We can safely assume, then, that there is some point X between A and B for which the power of this point with respect to c[1] is equal to the power of this point with respect to c[2].

The Chordal Theorem, which states that the radical axis of c[1] and c[2] is a line perpendicular to the line through the centers of the circles, tells us that the line through X which is perpendicular to AB is the radical axis of c[1] and c[2].

It can be shown that all circles orthogonal to c[1] and c[2] have centers that lie on the radical axis of c[1] and c[2].



From the diagram, for any point P on the radical axis, each tangent segment PM, PN, PO, and PQ are equal. A circle with center P and radius PM, then, will go through each point of tangency N, O, and Q, and by definition be orthogonal to both c[1] and c[2]. For any point P not on the radical axis, the tangents to c[1] and c[2] will not be equal, and therefore any circle with this center P not on the radical axis can not be orthogonal to both c[1] and c[2] concurrently.

The proof below relies on the following theorems/facts:

- Theorem: The radical axis of c1 and c2 is a line perpendicular to the line through the centers of the circles. *(from "The Power of a Point and Radical Axis" class notes)*

- To construct a circle orthogonal to 2 given circles, pick a point and find the inversion of the point with respect to each of the circles. The circle orthogonal to the 2 given circles goes through the original point and the 2 inversion points.

- If A is a point outside the circle and e is a circle which is orthogonal to c and centered at A, then if the radius of e is t, $p_c(A) = t^2$. *(from "The Power of a Point and Radical Axis" class notes)*

Note that the converse is also true: If $p_c(A) = t^2$, then the circle centered at A with radius t is orthogonal to circle c.

Concurrence of Radical Axes

<u>Theorem</u>: If a, b, and c are 3 circles and k, m, and n are the radical axes of b and c, c and a, a and b, respectively, then the 3 lines k, m, and n are either parallel or concurrent.

Proof:
Let the centers of circles a, b, and c be points A, B, and C respectively. Either A, B, and C must all be collinear, or any two of those points must be collinear. We will show that if the centers of the circles are all collinear, the lines k, m, and n must be parallel. Otherwise, k, m, and n meet at a single point.

First, consider the case where A, B, and C are collinear. Since the radical axis of two circles is a line perpendicular to the line through the centers of the circles, the radical axes k, m, and n are all perpendicular to the same line. Therefore, k, m, and n are parallel lines.

Figure 1, below, illustrates this scenario. The centers of circles a, b, and c are collinear. The center of a is in red; the center of b is in yellow; the center of c is in blue.

To find the radical axis of a and b, invert point N with respect to circle a to get point N'_a and with respect to circle b to get point N'_b. The circle orthogonal to a and b will contain the points N, N'_a, and N'_b. The orthogonal circle itself is not shown to avoid clutter in the diagram, but the center of that circle orthogonal to a and b is point O_N. Since we know that the radical axis of circles a and b is a line perpendicular to the line containing the centers of a and b, we can now construct the radical axis of a and b, n, as the line through O_N perpendicular to the line containing the centers of a and b.

Similarly, we invert point K with respect to circles b and c to find O_K in order to construct k, the radical axis of circles b and c.

In the same way, we invert point M with respect to circles a and c to find O_M and construct m, the radical axis of circles a and c.

Each of the lines n, m, and k are perpendicular to the same line (since the radical axis of 2 circles is a line perpendicular to the line through the centers of the 2 circles) and are therefore parallel.

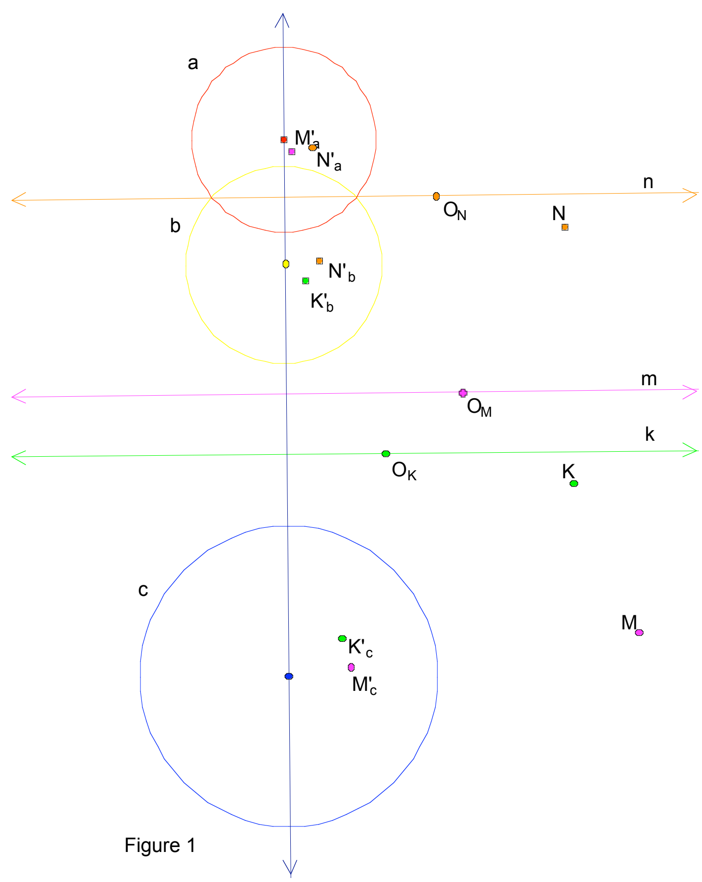


Figure 1

Suppose *A*, *B*, and *C* are not all collinear. Since *A*, *B*, and *C* all lie in the same plane, any two of *A*, *B*, and *C* must be collinear. Since the radical axis of two circles is perpendicular to the line containing the centers of the circles, we know that at least two of *k*, *m*, and *n* meet at a point.

Consider the line containing *A* and *B* and the line containing *B* and *C*. The radical axes of *a* and *b* and of *b* and *c* are perpendicular to two different lines that are not parallel or equal. Therefore, *n* and *k* are concurrent. Let *X* be the point where they meet.

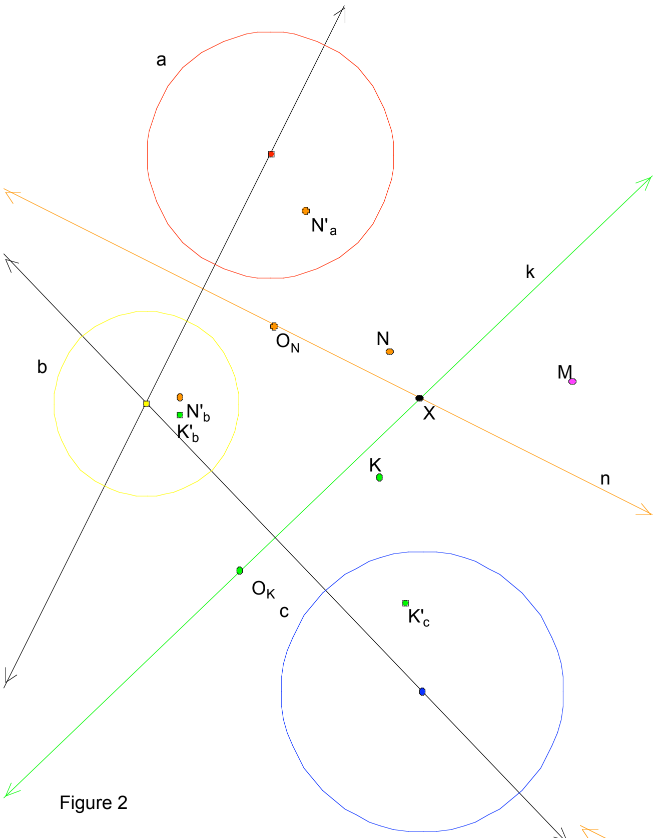


Figure 2

We know that, given a point *A* outside a circle *c*, and a circle *e* orthogonal to *c* centered at *A* with radius *t*, $p_c(A) = t^2$.

Since *X* lies on the radical axes *n* and *k*, there is a circle centered at *X* that is orthogonal to *a*, *b*, and *c*. Let *r* be the radius of the orthogonal circle centered at *X*. From the fact that *X* is on *n*, we know that $p_a(X) = p_b(X) = r^2$. Similarly, since *X* is on *k*, $p_b(X) = p_c(X) = r^2$. Therefore, by transitivity, $p_a(X) = p_c(X) = r^2$. This implies that the circle centered at *X* with radius *r* is orthogonal to *a* and *c*, which means that *X* is on the radical axis of *a* and *c*. Therefore, *m* must also go through *X*.

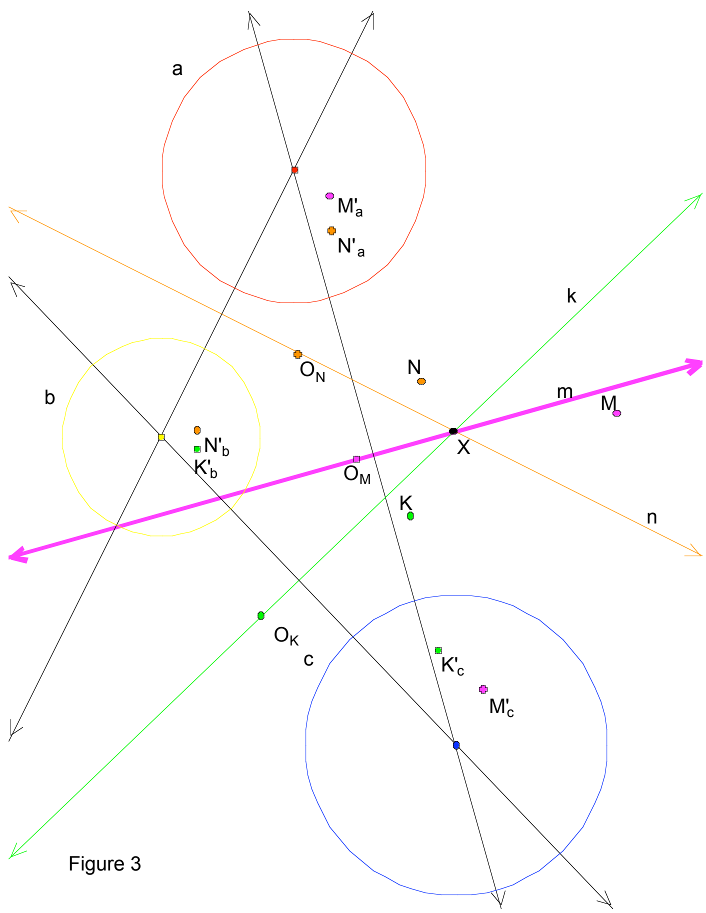
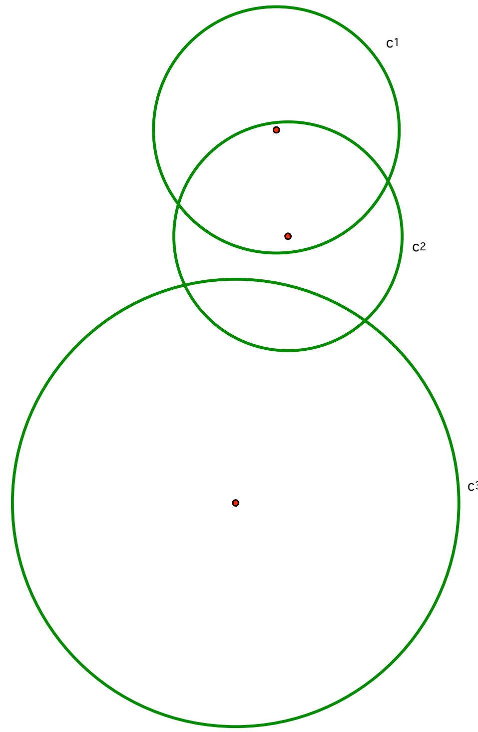


Figure 3

Therefore, there is either a single point where $p_a(X) = p_b(X) = p_c(X)$, or that point X does not exist. *k*, *m*, and *n* are either concurrent or parallel.

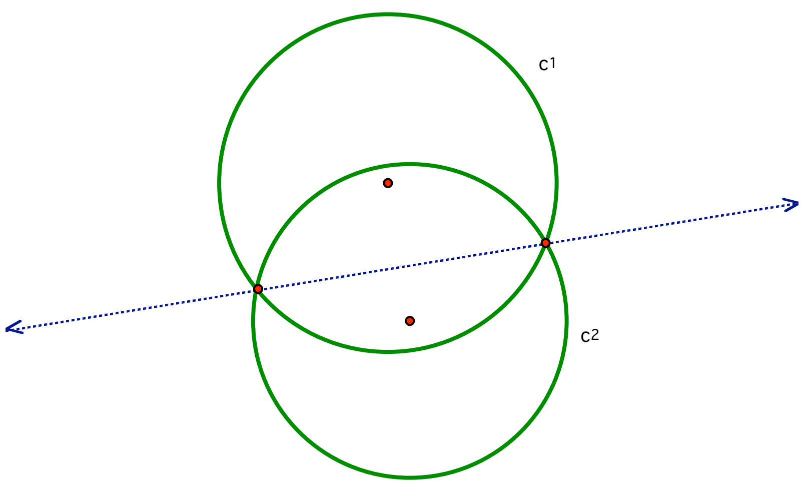
References
Math 445 Class Notes: [The Power of a Point and Radical Axis]

Now let's see how what we've learned about radical axes can be applied to solve a difficult problem. How would you construct a circle that is orthogonal to 3 given circles: c1, c2, and c3, as seen in the figure to the above?

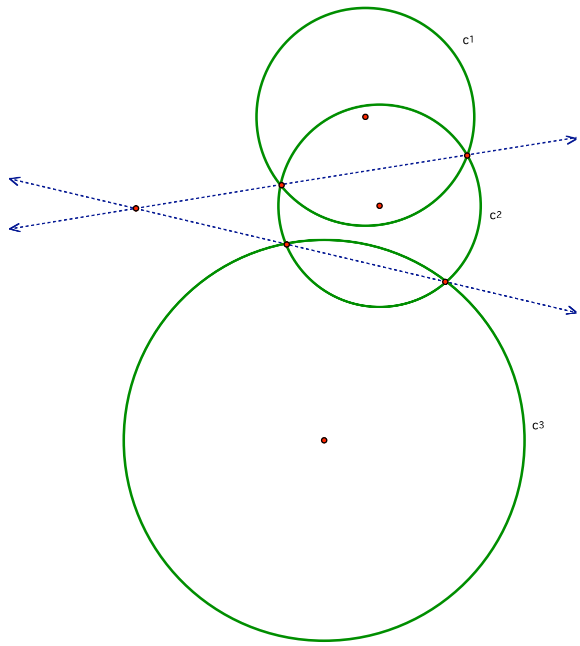


To start off with, look at two of the circles and ignore the third. Look at c1 and c2 for example, in the figure below. How would you find their radical axis? If you don't recall how, look back a few pages to the section on radical axes. Go ahead and construct the radical axis now.

What do you know about circles centered on the radical axis? That's right, you can always construct a circle that is orthogonal to the axis' two circles. What's more, there are an infinite number of circles that are orthogonal to c1 and c2, and their centers lie on a line—the radical axis! How would you construct an orthogonal circle to both c1 and c2? (Hint: first construct the tangents).



Now let's add the third circle (c3) back in. We can build on the work you've already done. Construct the radical axis of a different pair of circles, say c2 and c3. Where are the circles centered that are orthogonal to both c2 and c3? That's right, on their radical axis! Is there a point where you could center a circle that is orthogonal to c1, c2, and c3? Is there a point whose power is equal in all three circles? Construct the circle that is orthogonal to c1, c2, and c3 now.



Will there always be such a point to center the circle on? Explain.

There is always at least one point. Since the radical axes are lines, they either intersect or are parallel. So if the radical axes intersect, then there is one such point. But what if the radical axes are parallel? Do they ever meet? Yes! They meet at infinity! So the circle that is tangent to all three circles is the one with infinite radius (center at infinity), and looks exactly like a line that passes right through the centers of the 3 circles. Can you think of a case when the radical axes would be the same line? How many circles could you construct that are orthogonal to c1, c2, and c3 if the radical axes coincide?