Class Meeting #03

Axioms for Geometry

The B&B book uses an informal version of some rigorous axioms developed by G.D. Birkhoff, one of the first American mathematicians to achieve an international reputation for his research in differential equations and other areas. The hallmark of Birkhoff's axioms is that he assumes we already have the real numbers at hand and uses them to handle some technical problems of "betweenness" and intersections in geometry. In Euclid, the approach is the opposite one; the theory of number is developed from geometry.

Undefined Terms

Point, Line, On, Through, Distance, Angle Measure

Axiom 1. Ruler Axiom (B&B Principle 1)

The points on any straight line can be numbered so that number differences measure distances.

Comments: If you change units (e.g., from inches to centimeters) then you change the numerical value of the distance. Even with the same units, there are an infinite number of choices of ruler on each line. For example, you can have points A, B, C and some examples of rulers. Check that the first 3 give the same distance measures for AB, BC, AC. Tell what numbers in the missing cells will be consistent with these distances.

Points	**A**	**B**	**C**
Ruler 1 numbers	0	2	5
Ruler 2 numbers	-17	-15	-12
Ruler 3 numbers	4	2	-1
Ruler 4 numbers	1012		1017
Ruler 5 numbers		-32	-35

Consequences.

Corollary 1. If three points A, B, C are on a line and the distance AC is the largest among AB, AC, BC, then AB + BC = AC. In this case, if the rule associates points A, B, C to numbers a, b, c, then either $a > b > c$ or $a < b < c$. We say that B is ***between*** A and C (definition). The set of all points between A and C (plus A and C themselves) is called the **segment** AC (definition).

Corollary 2. Given a point A on a line and a distance $d > 0$, there are exactly two points on the line whose distance from A is d.

Corollary 3. Given two points A and B on a line, there is exactly one point M on the line which is equidistant from A and B. M is called the midpoint of segment AB.

Axiom 2. Two Point Axiom (B&B Principle 2)

There is one and only on straight line through two given points.

Consequences.

Two lines can have 0 points in common (disjoint lines) or intersect in one point (intersecting lines). If m and n are lines with two or more points in common, then m = n by the axiom.

Definition: Half-Lines or Rays

Let P be a point on a line. If a ruler assigns number p to point P, then consider the set of points U (with ruler number u) for which $p < u$. This is one of the half-lines with A as endpoint. The other half line is the set of V for which $p > v$. We can specify which of the two half-lines we mean by specifying a point Q on the half-line. For example, the half-line (or ray) PQ is the set of points Z on the line for which P is not between Q and Z. In other words, either Z is on segment PQ or Q is on segment PZ. Another way of saying this uses the ruler. If the ruler numbers of Q and Z are q and z, then if $q > p$, the half-line PQ is the set of Z for which $z > p$. If $q < p$, then half-line PQ is the set of Z with $z < p$.

Example 1: If we choose a ruler that assigns number 0 to P and number 1 to Q, then the points on the half-line PQ are the points Z with nonnegative ruler numbers.

Example 2: If we choose a ruler that assigns number 5 to P and number 2 to Q, then the points on the half-line PQ are the points Z with ruler numbers $z < 5$.

Definition: If point P is on a given line m, P is the end-point of two half-lines on m. These half-lines are called **opposite half-lines**.

Definition: Given non-collinear points V, A, B, the **angle AVB** is the figure consisting of the two half-lines VA and VB. We call a **straight angle** a figure consisting of opposite half-lines VA and VB.

Axiom 3. Protractor Axiom (B&B Principle 3)

All half-lines having the same end-point can be numbered so that number differences measure angles.

	Number of VA	Number of VB	Measure AVB
Protractor 1	0	50	50
Protractor 2	120	170	50
Protractor 3	20	-30	50
Protractor 4	20	330	310??

Axiom 4. Straight Angle Measure (B&B Principle 4)

All straight angles have the same measure.

Comment: The measure is most often chosen to be 180 degrees or π radians.

Definitions:

- Two angles AOB and BOC are a **linear pair** if OA and OC are opposite half-lines.
- Two angles are **supplementary** if they are congruent to the two angles of a linear pair. In other words, the sum of their angle measures equals the measure of a straight angle.
- Angle AOB and angle COD are **vertical angles** if OA and OC are opposite half-lines and also OB and OD are opposite half-lines. In other words, the lines AC and BD intersect at O.
- Given an angle AOB, let OA' be the opposite ray of OA, so that AOB and BOA' are a linear pair. If the angle AOB is congruent to BOA', the angle AOB is called a **right angle**.

Consequences

- Vertical angles are congruent.
- If two angles are congruent but also supplementary, then the angles are right angles.

Axiom 5. Side-Angle-Side [Congruence] (special case of B&B Principle 5)

Two triangles are congruent if an angle of one equals an angle of the other and the sides including the angles in one are equal to the sides including the angle in the other.

Comment:

Please note that this is an Axiom about congruence, whereas B&B jump right into similarity. We will introduce the full B&B axiom a bit later.