Rotations in 3-space: Rotation matrices and rotations of a cube

This project develops the matrix of any rotation in 3-space, starting with the simplest rotations. Given an axis of rotation and an angle, the goal is to find the 3x3 matrix, which rotates points (and objects) by this angle with this axis.

Prerequisites:

- Dot and cross product.
- Matrix of a linear transformation
- Recall $v \bullet w = 0$ means v and w are perpendicular (orthogonal).
- (Optional) Make or find a cardboard cube or wooden block, etc., and label the corners with the 8 coordinates $(\pm 1, \pm 1, \pm 1)$.

The Simplest Model case: Rotation around the x_3-axis

Visualize the x_1, x_2, x_3 axes in R^3 with unit vectors e_1, e_2, e_3 on each axis.

Matrices of rotations around the e_3 axis

1. Suppose N = Rotation by 90 degrees around the x_3-axis:
   Then $N(e_3) = e_3$, $N(e_1) = e_2$ and $N(e_2) = - e_1$.
   What is A = matrix N in this case?

2. Suppose T = Rotation by angle t around the x_3-axis.
   Then $T(e_3) = e_3$, $T(e_1) = (\cos t)e_1 + (\sin t)e_2$ and $T(e_2) = - (\sin t)e_1 + (\cos t)e_2$.
   What is B = matrix T in this case?

[Note: The direction of rotation is chosen so that it appears counter-clockwise when viewed with the eye located on the positive e_3-axis looking "down" towards 0.]

Check using the Cube

- Use N to rotate the cube with corners at the points $(\pm 1, \pm 1, \pm 1)$. This rotation should rotate the cube into itself. Take the rotation matrix A and check that each of the 8 vertices does in fact transform to another one of the vertices.
- Use T to rotate the cube. For what 4 angles t does T rotate the cube into itself?
- Take the rotation matrix B for these choices of t and check that each of the 8 vertices does in fact transform to another one of the vertices.

Analyzing Rotation around the x_3-axis using dot and cross product

Define the transformation $P(v) = (e_3 \cdot v)e_3$, using the dot product.

- Verify that P is a linear transformation with $P(e_3) = e_3$ but $P(e_1) = P(e_2) = 0$.

  What is C = matrix of transformation P?

- Observe that for v on the axis of rotation (= span e_3), P agrees with the rotation T but P is zero on the plane of vectors perpendicular to the axis of rotation (= span$\{e_1, e_2\}$).

Define the transformation $S(v) = e_3 \times v$, the cross product.

- Verify that S is a linear transformation with $S(e_3) = 0$ but $S(e_1) = e_2$ and $S(e_2) = - e_1$.

  What is D = matrix of transformation S?

- Observe that for v in the plane of vectors perpendicular to the axis of rotation, S agrees with the rotation N but S is zero on the axis of rotation.

Matrix Sum = Rotation Matrix

- Check that C + D = A. Does this make sense for v on the axis of rotation and also on the plane of vectors perpendicular to the axis of rotation?
- Check that $S^2(v) = 0$ when v is on the axis of rotation but $= -v$ for v perpendicular to the axis of rotation.
- What is the matrix E of S^2?
- Check that the transformation $U(v) = P(v) + (\sin t)S(v) - (\cos t)S^2(v)$ is the same as T by checking that they are equal on the 3 vectors e_1, e_2, e_3.
- This means matrix $B = C + (\sin t)D - (\cos t)E$. Check this.

The Rotation Matrix in the General Case

In the general case the same method applies for any axis. Choose a vector f on the axis of rotation. If f does not have length 1, divide f by its length so that it does have length 1.

For example, let $f = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Let $P(v) = (f \cdot v)f$ and let $S(v) = f \times v$ and $S^2(v) = f \times (f \times v)$.
Write the matrix of P, of S, of S^2 in this case.

Then let $R_f(v) = P(v) + (\sin t)S(v) - (\cos t)S^2(v)$. This transformation is the rotation by angle t with axis f. The matrix of R_f is the sum of the matrices of P and S.

[Note: The *direction* of rotation in this formula is chosen so that it appears counter-clockwise when viewed with the eye located on the positive f-axis looking "down" towards 0.]

What is matrix F = matrix of R_f with angle t = 120 degrees = $2\pi/3$ radians.

Check using the Cube

Use matrix F to rotate the cube with corners at the points (±1, ±1, ±1). This rotation should rotate the cube into itself. Take the rotation matrix of R_f and check that each of the 8 vertices does in fact transform to another one of the vertices.

Combining Rotations of the Cube

Use the same method to find the rotation R_g by 120 degrees = $2\pi/3$ radians but with axis

$$g = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

What is the matrix G = matrix of R_g with angle t = 120 degrees = $2\pi/3$ radians?

- Check that matrix G also rotates the cube into itself.

Product GF may provide a surprise

What is matrix GF = matrix of composition of the two previous rotations?

- Check that GF rotates the cube into itself. (This would be a good time to use your cardboard cube to see what is going on.)
- What is the axis of rotation of GF?
- What is the angle of rotation of GF? Do the two angles of F and G add up to the angle of GF?

More Rotations

Choose another axis vector h of length 1.

What is H = matrix of R_h with angle 45 degrees?

- Check that H(h) = h. (Why should this be true?)
- Also check that $H^4 = -I$. (Why should this be true?)

Extra – the General Case

Suppose that u = (a, b, c) is a vector of length 1, so $a^2+b^2+c^2 = 1$.

- Find the matrix R_u for rotation angle = 90 degrees.
- Find the matrix R_u for rotation angle t (i.e., the general formula).