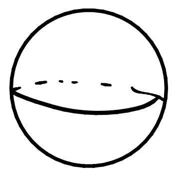
Abelian Sheaf Cohomology

5 June, 2019 07:50 PM

Example Consider the real valued functions on $\mathbb{S}^2 \subset \mathbb{R}^3$



To define a function on $\mathbb{S}^2$, we define a function on $\mathbb{R}^3$ and then require it to respect the equation of $\mathbb{S}^2$. In other words, we are

$$f \in \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$$

For any open subset U of $\mathbb{S}^2$, we can similarly define the functions on U. This data (open subset and real valued functions) is an example of a sheaf.

Def^n A presheaf of abelian groups on a topological space X is a functor

$$\mathcal{F} : Op(X)^{op} \longrightarrow \mathcal{A}b$$

Def^n A sheaf is a presheaf such that the following conditions are satisfied:

(i) **Locality** If $\{U_i\}$ is an open cover of an open set U and if $s, t \in \mathcal{F}(U)$ s.t. $s|_{U_i} = t|_{U_i}$ $\forall i$ then $s = t$

(ii) **Gluing** If $\{U_i\}$ is an open cover of an open set U and if for each i $\exists$ $s_i \in \mathcal{F}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_j \cap U_i}$ $\forall i,j$ then $\exists$ $s \in \mathcal{F}(U)$ s.t. $s|_{U_i} = s_i$ $\forall i$

Def^n A sheaf is a presheaf s.t. for any open cover $\{U_i\}$ of an open set U, the following diagram is an equalizer:

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

Note $U_i \cap U_j = U_i \times_U U_j$ in $Top(X)$

Def^n Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X.
A morphism $\varphi: \mathcal{F} \to \mathcal{G}$ is a morphism
$\varphi_U: \mathcal{F}(U) \to \mathcal{G}(U)$
for every open $U \subset X$ s.t. for any open $V \subset U$,

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\rho_{V,U}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}(U) \arrow[d, "\rho_{V,U}"] \\ \mathcal{F}(V) \arrow[r, "\varphi_V"] & \mathcal{G}(V) \end{tikzcd}$$

We say φ is injective (resp. surj) if $\forall x \in X$

$$\varphi_x : \operatorname{colim}_{U \ni x} \mathcal{F}(U) \to \operatorname{colim}_{U \ni x} \mathcal{G}(U)$$

is injective (resp. surj) (alt. use mono- & epimorphisms)

Claim Sheaves on X form an abelian category

Note that every sheaf is a presheaf. For any presheaf $\mathcal{F}$, we can associate a sheaf $\mathcal{F}^+$ by sheafifying $\mathcal{F}$. More concretely, there is a functor

$$(\ )^+ : \mathrm{PreSh}_X \to \mathrm{Sh}_X$$

such that $(\ )^+$ is left adjoint to the forgetful functor $\mathrm{Sh}_X \to \mathrm{PreSh}_X$

Def^n The global sections functor is a functor

$$\Gamma(X,-): \mathrm{Sh}_X \to \mathcal{Ab}$$

defined $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$

Claim $\Gamma(X,-)$ is left exact but not right exact

Left Exact $0 \to \mathcal{F} \to \mathcal{G} \Rightarrow$

$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \quad \forall$ open $U \subset X$

Not Right Exact let $X = \mathbb{C} \setminus \{0\}$

$\mathcal{O}$: holomorphic functions on X

$\mathcal{O}^*$ = $\{ f \in \mathcal{O} \mid f$ has no zeros$\}$

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

$$0 \to \Gamma(X, \underline{\mathbb{Z}}) \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathcal{O}^*) \to 0$$

$$\quad \mathbb{Z} \qquad\qquad \mathcal{O} \qquad\qquad \mathcal{O}^*$$

$1 \in \mathcal{O}^*$ but 1 is not in the image of map

Thm (Grothendieck 1957) Sh_X has enough injectives

We can define sheaf cohomology on X as the right derived functors of $\Gamma(X, -)$

Sheaf cohomology on topological spaces seems perfectly interesting until you remember that algebraic geometers use the Zariski topology. If you have a variety X, we can make X into a topological space by declaring the closed subsets of X to correspond to prime ideals containing the equations for X. This topology is too coarse for interesting invariants, in fact a huge number of cohomology groups vanish for small n. So algebraic geometers searched for a new topology to use in more general settings and came up with something that isn't really a topology applied to something which is not a topological space.