The Right Derived Functors of the Inverse Limit and Kernel Functors

Isaac Boekelheide

June 12, 2019

In these notes we discuss the right derived functors of the inverse limit functor. Weibel defines a collection of functors and then shows that they give the right derived functors to the inverse limit, but it is not transparent why he defines the collection this way and why it gives rise to the right derived functors to the inverse limit. Our goal is to motivate this definition and see why it is a natural construction.

For the entirety of the notes, let $\mathcal{A}$ be the category **Ab** of abelian groups, or any abelian category which satisfies the following additional axiom:

(AB4*): $\mathcal{A}$ is complete and the product of any collection of epimorphisms is an epimorphism.

Complete means that the product of any small diagram exists. We will point out where we use these extra assumptions. We further assume that $\mathcal{A}$ has enough injectives.

We begin with the collection of functors which Weibel defines as

Definition. Given a tower $\{A_i\}$ in $\mathcal{A}$, consider the map

$$\Delta : \prod_{i \in \mathcal{I}} A_i \to \prod_{i \in \mathcal{I}} A_i$$

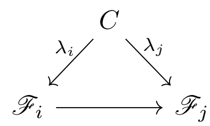
defined by $\Delta = \mathrm{Id} - \alpha$, where $\mathrm{Id} = \mathrm{Id}_{\prod A_i}$ for ease of notation. Define

$$\varprojlim{}^n A_i = \begin{cases} \varprojlim A_i & n = 0 \\ \text{coker } \Delta & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Weibel proves these are the right derived functors of the inverse limit, but does little to motivate the definition. Our goal is to illuminate this definition, and why it ends up giving us the right derived functors of the inverse limit.

First, we review the concept of the limit of a diagram in a category. We first introduce the notion of a cone over a diagram.

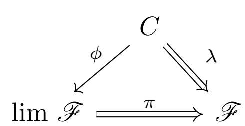
Definition (Cone)**.** A cone over a diagram $\mathscr{F} : \mathcal{I} \to \mathcal{A}$ is an object C along with maps $\lambda_i : C \to \mathscr{F}_i$ for every i in I such that for every morphism $i \to j$ in $\mathcal{I}$, the following diagram commutes:



We say that the λ_i's are the legs of the cone. For notation, we say $\lambda : C \Rightarrow \mathscr{F}$ is a cone over $\mathscr{F}$.

Remark. A convenient way to think about a cone over a diagram is to think of C as a constant diagram over $\mathcal{I}$ with each morphism the identity morphism on C. Then λ is nothing more than a natural transformation from the constant diagram in C to the diagram $\mathscr{F}$. With this understanding, we can see that the notation $\lambda : C \Rightarrow \mathscr{F}$ aligns with the notation for a natural transformation between two diagrams, with the legs of the cone as its components.

Definition (Limit of a Diagram)**.** A limit to a diagram $\mathscr{F} : \mathcal{I} \to \mathcal{A}$ is a cone $\pi : \lim \mathscr{F} \Rightarrow \mathscr{F}$ with the following universal property: given any other cone $\lambda : C \Rightarrow \mathscr{F}$ there exists a unique morphism $\phi : C \to \lim \mathscr{F}$ such that the following diagram commutes:



This notation shows that the legs of the cone λ must factor through π, i.e., for all i in $\mathcal{I}$, we have that $\lambda_i = \pi_i \circ \phi$

Now, let $\mathcal{A}^{\mathcal{I}}$ be the category of diagrams of shape $\mathcal{I}$ in $\mathcal{A}$ with morphisms as natural transformations. If $\mathcal{A}$ is abelian, then $\mathcal{A}^{\mathcal{I}}$ is abelian (Weibel 1.6.4). Given that $\mathcal{A}$ has all limits of diagrams of shape $\mathcal{I}$, taking the limit of a diagram is a functor:

$$\lim : \mathcal{A}^{\mathcal{I}} \to \mathcal{A}$$

That taking the limit is functorial follows from the universal property of the limit (check this). Thus, when

$$\mathcal{I} : \cdots \to 2 \to 1 \to 0,$$

the inverse limit, $\varprojlim$, is a functor. We shall show that all inverse limits exist for all diagrams of shape $\mathcal{I}$. It is standard to call the objects of $\mathcal{A}^{\mathcal{I}}$ towers of objects in $\mathcal{A}$, or when $\mathcal{A} = \mathbf{AB}$, towers of abelian groups.

It is a fact that the limit of any diagram of any shape exists in any complete abelian category, which is the first time we invoke (AB4*). For our needs however, we will only deal with the $\mathcal{I}$ given above. Indeed, let $\{A_i\}_{i\in\mathcal{I}}$ be a diagram of shape $\mathcal{I}$ in $\mathcal{A}$. Our first goal is to use the product to represent the entire diagram with the least number of objects, as follows:

Because $\mathcal{A}$ is complete, the product indexed over $\mathcal{I}$

$$\prod_{i\in\mathcal{I}} A_i$$

exists. In some sense, the product remembers all objects in diagram, because a map into $\prod A_i$ is just a map into each object of the diagram. Therefore, any cone over $\{A_i\}$ has a map into $\prod A_i$. However, by taking the product, we lose the information about the morphisms in the diagram, so we create a map between the product and itself which remembers these morphisms. A map into a product is fully determined by maps into its components, so we create a map

$$\alpha : \prod_{i\in\mathcal{I}} A_i \to \prod_{i\in\mathcal{I}} A_i$$

depending on the maps in the diagram. This map is defined as follows. For every morphism $\alpha_{i+1} : A_{i+1} \to A_i$ in the the original diagram, the following diagram commutes:

$$\begin{array}{ccc} \prod_{i\in\mathcal{I}} A_i & \xrightarrow{\alpha} & \prod_{i\in\mathcal{I}} A_i \\ {\scriptstyle \pi_{i+1}}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ A_{i+1} & \xrightarrow[\alpha_{i+1}]{} & A_i \end{array}$$

For ease of notation, let $\mathrm{Id} = \mathrm{Id}_{\prod A_i}$. We claim the kernel of $\mathrm{Id} - \alpha$ is the limit of $\{A_i\}$. Indeed, let $\lambda : C \Rightarrow \{A_i\}$ denote a cone over $\{A_i\}$. The legs of λ define a map into the product, which we also denote by λ:

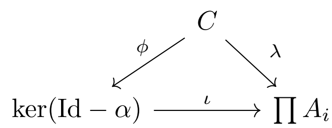
$$\begin{array}{ccc} C & \xrightarrow{\lambda} & \prod_i A_j \\ & {\scriptstyle \lambda_i}\searrow & \downarrow{\scriptstyle \pi_i} \\ & & A_i \end{array}$$

We further have that for each i, $\lambda_i = \alpha_{i+1}\lambda_{i+1}$. In terms of our maps between the products, this means that the components of the maps given by the diagram

$$\begin{array}{ccccc} C & \xrightarrow{\lambda} & \prod_{i\in\mathcal{I}} A_i & \xrightarrow{\alpha} & \prod_{i\in\mathcal{I}} A_i \\ & {\scriptstyle \lambda_{i+1}}\searrow & \downarrow{\scriptstyle \pi_{i+1}} & & \downarrow{\scriptstyle \pi_i} \\ & & A_{i+1} & \xrightarrow[\alpha_{i+1}]{} & A_i \end{array}$$

are the same as the diagram above it, so $\lambda = \alpha\lambda$, i.e, that $(\mathrm{Id} - \alpha)\lambda = 0$. (Note the appearance of $\Delta = \mathrm{Id} - \alpha$ from Weibels definition). Conversely, this information is enough to specify a cone. Given a map $\lambda : C \to \prod A_i$ such that $(\mathrm{Id} - \alpha)\lambda = 0$, this says for every i, that $\lambda_i = \alpha_{i+1}\lambda_{i+1}$.

Now, this means that λ factors uniquely through $\ker(\mathrm{Id}-\alpha)$, and $\ker(\mathrm{Id}-\alpha)$ is also a cone above $\{A_i\}$ with legs ι. i.e. there exists a unique morphism $\phi : C \to \ker(\mathrm{Id} - \alpha)$ such that the following diagram commutes:



Picking the component for any i, we get that $\lambda_i = \iota_i\phi$, and ϕ is unique by the uniqueness above, so we see that $\ker(\mathrm{Id} - \alpha)$ is a limit, or inverse limit, of the diagram $\{A_i\}$. Thus, we see that having products is sufficient to have all limits in an abelian category.

This is actually a general categorical result, even in a non-abelian category. Any limit can be represented as the equalizer of a map between products depending on the diagram, so to check if a category has limits for all small diagrams (is complete), it is sufficient to check that it has equalizers and small products. A detailed discussion can be read in "Category Theory in Context" by Emily Riehl. In particular, the statement is given for sets in Theorem 3.2.14 and dual is stated in Theorem 3.4.13. Because we can subtract morphisms in an abelian category, any equalizer can be represented as a kernel, which always exists. Thus, an abelian category is complete if and only if it has all small products, which is where the terminology aligns for both cases.

We now discuss why limit functors are left exact. The universal property of limits makes the limit functor right adjoint to the constant diagram functor, and is thus left exact. This follows from the fact that a natural transformation out of the apex of a cone, which is a constant diagram of an object in $\mathcal{A}$, corresponds uniquely to a map from that object into the limit of the diagram. Refer to Wiebel 2.6.7, 2.6.9, and Exercise 2.6.4 for a more in depth discussion. Because the limit functor is left exact, we can compute its right derived functors, given that $\mathcal{A}^{\mathcal{I}}$ has enough injectives, but if $\mathcal{A}$ has enough injectives, then so does $\mathcal{A}^{\mathcal{I}}$ (Weibel Example 2.3.13).

We now return to Weibel's definition:

Definition. Given a tower $\{A_i\}$ in $\mathcal{A}$, consider the map

$$\Delta : \prod_{i \in \mathcal{I}} A_i \to \prod_{i \in \mathcal{I}} A_i$$

defined by $\Delta = \mathrm{Id} - \alpha$. Define

$$\varprojlim{}^n A_i = \begin{cases} \varprojlim A_i & n = 0 \\ \operatorname{coker} \Delta & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

After the above discussion, it is more clear where this definition is coming from. Note that we already showed that $\varprojlim A_i = \ker \Delta$, so we can look at the definition as a collection of functors T^n acting on a morphism Δ

$$T^n(\Delta) = \begin{cases} \ker \Delta & n = 0 \\ \text{coker } \Delta & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now, this is starting to suggest another pattern, that in some sense the cokernel functor provides the first right derived functor of the kernel functor, and the other right derived functors are 0. We make this precise by defining the arrow category of $\mathcal{A}$, which we denote $\text{Arr}(\mathcal{A})$. The objects are morphisms in $\mathcal{A}$ and the morphisms are commuting squares between morphisms. For example, given object morphisms, $\phi : A \to B$ and $\psi : C \to D$, a commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow[\psi]{} & D \end{array}$$

is a morphism between ϕ and ψ. Given that $\mathcal{A}$ is abelian, the arrow category is abelian, because we can view the arrow category as a diagram category $\mathcal{A}^{\mathcal{J}}$, where

$$\mathcal{J} : 1 \to 2.$$

We see that

$$\ker, \text{coker } : \text{Arr}(\mathcal{A}) \to \mathcal{A}$$

are two functors defined on $\text{Arr}(\mathcal{A})$. The kernel is a limit functor, so it is left exact, and thus we can compute its right derived functors. We provide further motivation for the cokernel being the right derived functor of kernel because we already have a tool which sends short exact sequences of morphisms

$$0 \to f \to g \to h \to 0$$

to long exact sequences. Expanding these morphisms into commuting squares, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

with exact rows. We know that the snake lemma gives us a long exact sequence:

$$0 \to \ker f \to \ker g \to \ker h \to \text{coker } f \to \text{coker } g \to \text{coker } h \to 0.$$

This is no proof, but the snake lemma takes a short exact sequence in $\mathrm{Arr}(\mathcal{A})$ to a long exact sequence. Again we see that pattern that suggests the long exact sequence arising from the right derived functors of ker should be the long exact sequence given by the snake lemma!

Unfortunately, after the presentation, I was not able to show that these are the right derived functors of ker using tools we developed in this class, but it is somewhat straight forward using the techniques of cohomological δ-functors developed by Wiebel in section 2.1. I will not supply the proof, to keep these notes short, and instead provide a road map for those interested in looking further into this topic.

1. Show that if $\mathcal{A}$ has enough injectives, then $\mathrm{Arr}(\mathcal{A})$ has enough injectives taking the form of projections $I_1 \oplus I_2 \to I_1$, where I_1 and I_2 are injective objects of $\mathcal{A}$.

2. Use the snake lemma to show that

$$T^n(\Delta) = \begin{cases} \ker \Delta & n = 0 \\ \operatorname{coker} \Delta & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

   is a cohomological δ-functor (Weibel Definition 2.1.1)

3. Show that T^n is a universal δ-functor (Weibel Definition 2.1.4) by showing that T^n is effaceable for $n > 0$ (Weibel Exercise 2.4.5) by showing that every object morphism ϕ in $\mathrm{Arr}(\mathcal{A})$ has a monic commuting square towards an injective morphism of the type in step 1, and showing that T^n for $n > 0$ vanishes on injectives of this type.

4. Use the dual to Weibel Theorem 2.4.6 and 2.5.1 to see that right derived functors form a universal cohomological δ-functor.

5. Finally, prove that if two universal cohomological δ-functors T^n and S^n have $T^0 = S^0$, then T^n and S^n are isomorphic as δ-functors. Use this fact to show that T^n provides the right derived functors of the ker functor.

Now that we have the right derived functors of ker, notice that we haven't exactly shown that this gives the right derived functors of $\varprojlim$. The $\varprojlim$ functor acts on $\mathcal{A}^{\mathcal{I}}$, but ker acts on $\mathrm{Arr}(\mathcal{A})$, so how are the right derived functors of ker actually giving the right derived functors of $\varprojlim$ when the domain categories are different? We see that we have a functor

$$G : \mathcal{A}^{\mathcal{I}} \to \mathrm{Arr}(\mathcal{A})$$

$$\{A_i\} \mapsto \Delta : \prod_{i\in\mathcal{I}} A_i \to \prod_{i\in\mathcal{I}} A_i$$

which sends a natural transformation $f : \{A_i\} \to \{B_i\}$ to the commuting square

$$\begin{array}{ccc}
\prod_{i\in\mathcal{I}} A_i & \xrightarrow{\Delta} & \prod_{i\in\mathcal{I}} A_i \\
\downarrow & & \downarrow \\
\prod_{i\in\mathcal{I}} B_i & \xrightarrow[\Delta]{} & \prod_{i\in\mathcal{I}} B_i
\end{array}$$

where the diagonal arrows are given by the component maps

$$\begin{array}{ccc}
\prod_{i\in\mathcal{I}} A_i & \longrightarrow & \prod_{i\in\mathcal{I}} B_i \\
\downarrow & & \downarrow \\
A_i & \xrightarrow[f_i]{} & B_i
\end{array}$$

If we let $F = \ker$, then we see that $\varprojlim = FG$. Thus, by computing the right derived functors of ker and claiming it is the right derived functors of $\varprojlim$, we are essentially saying that

$$R^n(FG) = (R^nF) \circ G.$$

We can see that sufficient conditions for this to be true is if G is exact and sends injectives to F-acyclic objects. Indeed if this is the case, then given an injective resolution $I = \{I_i^m\}$ of $A = \{A_i\}$, i.e.,

$$0 \to \{A_i\} \to \{I_i^1\} \to \{I_i^2\} \to \cdots$$

Then, because G is exact and sends injectives to F-acyclic objects, we see that $G(I)$ is an F-acyclic resolution of $G(A)$, so

$$R^n(FG)(A) = H^n(FG(I)) = H^n(F(G(I))) = R^nF(G(A)) = (R^nF) \circ G(A).$$

Thus, it remains to show that G is exact and sends injectives to F-acyclic objects. Indeed, if we have a short exact sequence

$$0 \to \{A_i\} \to \{B_i\} \to \{C_i\} \to 0,$$

Then we need the commuting diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_{i\in\mathcal{I}} A_i & \longrightarrow & \prod_{i\in\mathcal{I}} B_i & \longrightarrow & \prod_{i\in\mathcal{I}} C_i & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} & & \downarrow \\
0 & \longrightarrow & \prod_{i\in\mathcal{I}} A_i & \longrightarrow & \prod_{i\in\mathcal{I}} B_i & \longrightarrow & \prod_{i\in\mathcal{I}} C_i & \longrightarrow & 0
\end{array}$$

This is where we finally use the remaining condition in axiom (AB4*). The product of a collection of epimorphisms is an epimorphism, so we actually get the exactness of the rows on the right. The exactness on the left follows because the product functor, being a limit, is left exact. Thus, we conclude that G is exact.

Now, to show that G sends injectives to F-acyclic objects, we first note that enough injective objects in $\mathcal{A}^{\mathcal{I}}$ are towers of the form

$$\cdots \to I_0 \oplus I_1 \oplus I_2 \to I_0 \oplus I_1 \to I_0$$

as one can check. These maps are epimorphisms, so G sends this to an epimorphism. We can see that epimorphisms are F-acyclic, because the only non-zero right derived functor for $F = \ker$ is coker , which is 0 on epimorphisms, so G sends injectives to F-acyclic objects. Thus, we see that $G(I)$ is an F-acyclic resolution, and we can conclude that

$$R^n(FG) = (R^n F) \circ G,$$

so we have fully provided the right derived functors to the inverse limit.