LIE ALGEBRA COHOMOLOGY

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1. Cohomology of Compact Lie Groups

First recall that a Lie group is a smooth manifold G that is also a group in the algebraic sense, with the property that the multiplication map and inversion map are both smooth. In particular, if we let $L_g : G \to G$ denote left multiplication by g, then the space of left-invariant vector fields (i.e. $X \in \mathfrak{X}(M)$ such that $(L_g)_*(X) = X$, or equivalently, $d(L_g)_h(X_h) = X_{gh}$) form a Lie algebra, which we shall denote by $\mathfrak{g}$. Specifically this means that $\mathfrak{g}$ is a vector space together with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying, for all $X, Y, Z \in \mathfrak{g}$:

(1) Bilinearity: for $a, b \in \mathbb{R}$:

$$[aX + bY, Z] = a[X, Y] + b[Y, Z]$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y]$$

(2) Antisymmetry:

$$[X, Y] = -[Y, X]$$

(3) Jacobi Identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

We call this map the Lie bracket of $\mathfrak{g}$.

Remark 1. It is worth noting that the evaluation map $\varepsilon : \mathfrak{g} \to T_eG$ (where $e \in G$ is the identity) given by $\varepsilon(X) = X_e$, is a natural vector space isomorphism, which will be important later on. See [4], proposition 8.37, for a proof of this result.

Example 2.

(1) The special orthogonal group of degree n, denoted by $SO(n)$ is the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of orthogonal matrices with determinant equal to 1. Its Lie algebra, which we shall denote by $\mathfrak{so}(n)$, consists of traceless $n \times n$ real matrices.

(2) Similarly, the special unitary group of degree n, denoted by $SU(n)$, consists of unitary matrices with determinant equal to 1, and its Lie algebra $\mathfrak{su}(n)$ consists of traceless skew-Hermitian matrices. It can be shown that $SO(3)$ and $SU(2)$ are not diffeomorphic since $SO(3)$ is not simply connected but $SU(2)$ is (in fact $SU(2)$ is the universal cover of $SO(3)$). However, $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic as Lie algebras, and both are isomorphic to $\mathbb{R}^3$ with the cross-product. Recall that if two simply-connected Lie groups have isomorphic Lie algebras, then the groups must have been isomorphic as well (see theorem 20.21 in [4]).

Now let $\Omega^n(G)$ denote the space of differential n-forms. We then say a differential form $\omega \in \Omega^*(G)$ is left-invariant if $L_g^*\omega = \omega$ for all $g \in G$ (where L_g^* is the pullback, more explicitly given by $(L_g^*\omega)_h(v) = \omega_{gh}(d(L_g)_h(v)$ for $v \in T_hG$). We then denote the space of left-invariant n-forms by $\Omega_L^n(G)$, which, by linearity of L_g^* form a subspace of $\Omega^n(G)$. By properties of pullbacks, $L_g^*d\omega = dL_g^*\omega$, and consequently $d(\Omega^n(G)) \subset \Omega^{n+1}(G)$. Consequently, we can view

$$0 \to \Omega_L^0(G) \xrightarrow{d} \Omega_L^1(G) \xrightarrow{d} \Omega_L^2(G) \xrightarrow{d} \dots$$

as a subcomplex of the de Rham complex

$$0 \to \Omega^0(G) \xrightarrow{d} \Omega^1(G) \xrightarrow{d} \Omega^2(G) \xrightarrow{d} \dots$$

Let us denote the cohomology of the first complex by $H^n_L(G)$ and the second by $H^n_{\mathrm{dR}}(G)$. Since pullbacks also distribute across wedge products, this implies that $H^*_L(G)$ has a ring structure induced by the wedge product, just as $H^*_{\mathrm{dR}}(G)$ does.

Theorem 3. *Suppose G is compact and connected, and let $\iota : \Omega^*_L(G) \to \Omega^*(G)$ denote the inclusion chain map. Then the induced map $\iota_* : H^*_L(G) \to H^*_{dR}(G)$ is a ring isomorphism.*

Proof. (Sketch) Define a map $I : \Omega^n(G) \to \Omega^n_L(G)$ by

$$I(\omega) = \int_G L_g^* \omega dg,$$

where dg denotes the normalized Haar measure of G (specifically this means that $\int_G dg = 1$; this is where compactness is important). Then I is linear and commutes with d and pullbacks. This immediately implies injectivity of ι_*, for if $\iota_*[\omega] = 0$, then $\omega = d\mu$ for some $\mu \in \Omega^{n-1}(G)$, and consequently

$$\omega = I(\omega) = I(d\mu) = d(I(\omega)),$$

and so $\omega \in d(\Omega^{n-1}_L(G)$, which implies $[\omega] = 0$.

To show surjectivity, let $[\omega] \in H^n_{\mathrm{dR}}(G)$ and let $\langle \cdot, \cdot \rangle : H^n_{\mathrm{dR}}(G) \times H_n(G, \mathbb{R}) \to \mathbb{R}$ denote the natural pairing given by integration (see theorem 18.14 in [4]). Then in particular this pairing is nondegenerate, and so if we can show that $\langle [\omega - I(\omega)], [Z] \rangle = 0$ for all n-cycles Z (i.e. closed n-submanifolds), then it will follow that $[\omega - I(\omega)] = 0$, and so $[\omega] = \iota_*[I(\omega)]$. To show this claim, first note that since G is connected $[Z] = [gZ]$ as homology classes (in particular, Z and gZ differ by a boundary): $Z - gZ = \partial Z'$ for some $(n-1)$-cycle Z'). Thus we compute the following:

$$\begin{aligned}
\int_Z \omega - I(\omega) &= \int_Z \omega - \int_Z \int_G L^* g \omega dg \\
&= \int_Z \omega - \int_G \int_Z L_g^* \omega dg \\
&= \int_Z \omega - \int_G \int_{gZ} \omega dg \\
&= \int_Z \omega - \int_G \int_Z \omega dg \\
&= \int_Z \omega - \left(\int_G dg \right) \left(\int_Z \omega \right) \\
&= \int_Z \omega - 1 \cdot \int_Z \omega \\
&= 0.
\end{aligned}$$

Thus ι_* is a ring isomorphism. □

With this result in hand, we can actually take this one step further. In particular, because every left invariant form is determined by its value at the identity, we are naturally led to the following result.

Theorem 4. *Let $\delta : (\Lambda^n \mathfrak{g})^* \to (\Lambda^{n+1} \mathfrak{g})^*$, where $(\Lambda^n \mathfrak{g})^* = \mathrm{Hom}_{\mathbb{R}}(\Lambda^n \mathfrak{g}, \mathbb{R})$ is the space of skew-symmetric n-linear forms on $\mathfrak{g}$, be given by, for $\alpha \in (\Lambda^n \mathfrak{g})^*$ and $X_1, \ldots, X_{n+1} \in \mathfrak{g}$,*

$$(\delta\alpha)(X_1|_e, \ldots, X_{n+1}|_e) = \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j]|_e, X_1|_e, \ldots, \widehat{X}_i|_e, \ldots, \widehat{X}_j|_e, \ldots X_{n+1}|_e).$$

Then the evaluation map $\varepsilon : \Omega^*_L(G) \to (\Lambda^* \mathfrak{g})^*$ give by $\varepsilon(\omega) = \omega_e$ is both a graded ring isomorphism and a chain isomorphism between the complex of left invariant forms and the complex

$$0 \to (\Lambda^0 \mathfrak{g})^* \xrightarrow{\delta} (\Lambda^1 \mathfrak{g})^* \xrightarrow{\delta} (\Lambda^3 \mathfrak{g})^* \xrightarrow{\delta} \ldots$$

Remark 5. Note that $(\Lambda^0\mathfrak{g})^* \cong \mathbb{R}$ and $(\Lambda^1\mathfrak{g})^* \cong \mathfrak{g}^*$.

Proof. (Sketch) First note that, as a restriction map, ε is clearly a ring homomorphism. To see that ε commutes with the corresponding differentials, and consequently is also a chain map, observe that if $\omega \in \Omega^n_L(G)$ and $X_1, \dots, X_{n+1} \in \mathfrak{g}$, then

$$\omega(X_1, \dots, X_n)(g) = \omega_g(X_1|_g, \dots, X_n|_g) = (L_g^*\omega)_e(X_1|_e, \dots, X_n|_e) = \omega_e(X_1|_e, \dots, X_n|_e),$$

which in particular tells us that $\omega(X_1, \dots, X_n)$ is constant, viewed as a real-valued function on G. As a result, we have that

$$\begin{aligned}
\varepsilon(d\omega)(X_1|_e, \dots, X_{n+1}|_e) &= (d\omega)_e(X_1|_e, \dots, X_{n+1}|_e) \\
&= d\omega(X_1, \dots, X_{n+1})(e) \\
&= \sum_{i=1}^{n} (-1)^i X_i(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{n+1})(e) \\
&\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots X_{n+1})(e) \\
&= \sum_{i=1}^{n} (-1)^i \cdot 0 + \sum_{i<j} (-1)^{i+j} \omega_e([X_i, X_j]|_e, X_1|_e, \dots, \widehat{X}_i|_e, \dots, \widehat{X}_j|_e, \dots X_{n+1}|_e) \\
&= \delta(\varepsilon(\omega))(X_1|_e, \dots, X_{n+1}|_e),
\end{aligned}$$

and so ε is a chain map. Lastly, it is bijective, for if $\alpha \in (\Lambda^n\mathfrak{g})^*$, then by left invariance there is a unique $\omega \in \Omega^n_L(G)$ satisfying $\omega_e = \alpha$, given by

$$\omega_g(X_1|_g, \dots, X_n|_g) = \alpha((L_{g^{-1}})_*(X_1|_g), \dots, (L_{g^{-1}})_*(X_n|_g)).$$

Thus, ε is a graded ring and a chain isomorphism, as claimed. □

Since the two complexes $\Omega^*_L(G)$ and $(\Lambda^*\mathfrak{g})^*$ are isomorphic, it immediately follows that their cohomology rings are as well. Thus, if we denote the cohomology ring of this latter complex by $H^*(\mathfrak{g})$, which we shall call the Lie algebra cohomolofy of $\mathfrak{g}$, then, for a compact, connected Lie group G, have the following sequence of isomorphisms

$$H^*_{\mathrm{dR}}(G) \cong H^*_L(G) \cong H^*(\mathfrak{g}).$$

This then immediately gives us the following result.

Corollary 6. *If G_1, G_2 are compact, connected Lie groups with corresponding Lie algebra's $\mathfrak{g}_1, \mathfrak{g}_2$ such that $\mathfrak{g}_1 \cong \mathfrak{g}_2$, then $H^*_{dR}(G_1) \cong H^*_{dR}(G_2)$*

Example 7. Let us again consider the groups $SO(3)$ and $SU(2)$. Since their Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic, the above corollary tells us that $H^*_{\mathrm{dR}}(SO(3)) \cong H^*_{\mathrm{dR}}(SU(2))$. This in turn tells us, by the de Rham theorem, that their singular cohomology rings (with real coefficients) are isomorphic as well: $H^*(SO(3), \mathbb{R}) \cong H^*(SU(2), \mathbb{R})$, and in particular, both are isomorphic to $\mathbb{R}[x]/(x^2)$. However, it is important to note that this is not necessarily true if some other coefficient had been used instead. For example, $H^2(SO(3), \mathbb{Z}) = \mathbb{Z}/2$, while $H^2(SU(2), \mathbb{Z}) = 0$.

2. Lie Algebras Over Arbitrary Rings

Definition 8. Let k be a fixed commutative ring. A nonassociative algebra A is a k-module equipped with a bilinear product $A \otimes_k A \to$. A Lie algebra $\mathfrak{g}$ is a nonassociative algebra whose product, written as $[x, y]$ called the **Lie Bracket**, satisfies for $x, y, z \in \mathfrak{g}$:

- (**Skew-symmetry**) $[x, x] = 0$ (and hence $[x, y] = -[y, x]$)
- (**Jacobi's Identity**) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Definition 9. An **ideal** of $\mathfrak{g}$ is a k-submodule $\mathfrak{h}$ such that $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$, that is for all $g \in \mathfrak{g}$ and $h \in \mathfrak{h}$, $[h, g] \in \mathfrak{h}$. The quotient $\mathfrak{g}/\mathfrak{h}$ inherits the structure of a Lie algebra.

Definition 10. There is a natural functor Lie : $Associative, k - alg \to Lie - Alg$, where Lie($A$) is the assocative algebra A with the lie bracket as the commutator.

Definition 11. (The Universal Enveloping Algebra) Let $\mathfrak{g}$ be a lie algebra. We define $U\mathfrak{g}$ as

$$U\mathfrak{g} = T(\mathfrak{g})/\langle [x, y] - x \otimes y + y \otimes x \rangle.$$

2.1. $\mathfrak{g}$-modules.

Definition 12. Let $\mathfrak{g}$ be a Lie algebra over k. A (left) $\mathfrak{g}$-module M is a k-module equpped with a k-bilinear product $\mathfrak{g} \otimes_k M \to M$ (written $x \otimes m \to xm$) such that

$$[x, y]m = x(ym) - y(xm) \text{ for all } x, y \in \mathfrak{g} \text{ and } m \in M$$

Example 13.
1. Any Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-module over its self by the action $x.m = [x, m]$ (the Jacobi Identity makes sure the above relationsip is satsified).
2. A trivial $\mathfrak{g}$-module is a k-module M on which $\mathfrak{g}$ acts: $xm = 0$ for all $x \in \mathfrak{g}$, and $m \in M$. For example, we can make k in to a $\mathfrak{g}$-module with the trivial action.

Definition 14. A (left) $\mathfrak{g}$-module homomorphism $f : M \to N$ is a k-module map that is $f(xm) = xf(m)$, and denote $\mathrm{Hom}_{\mathfrak{g}}(M, N)$ to be the set of all $\mathfrak{g}$-module homomorphism.

Functors between k-Mod and $\mathfrak{g}$-Mod.

Definition 15.
- (Trivial $\mathfrak{g}$-module functor) Takes k-modules to $\mathfrak{g}$-modules with the trivial action.
- $(-)^{\mathfrak{g}} : \mathfrak{g} - \mathsf{Mod} \to k - \mathsf{Mod}$ is the functor that takes $\mathfrak{g}$-module M to

  $$M^{\mathfrak{g}} = \{m \in M \mid xm = 0 \; \forall x \in \mathfrak{g}\}.$$

  This is called the invariant submodule of M. If we treat k as the trivial $\mathfrak{g}$-module, then we have the isomorphism

  $$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}}(k, M) &\to M^{\mathfrak{g}} \\ \varphi &\to \varphi(1) \\ \varphi_m &\leftarrow m \end{aligned}$$

  where $\varphi_m(1) = m$.
- $(-)_{\mathfrak{g}} : \mathfrak{g} - \mathsf{Mod} \to k - \mathsf{Mod}$ is the functor that takes $\mathfrak{g}$-module M to

  $$M^{\mathfrak{g}} = M/\mathfrak{g}M,$$

  and this is called the coinvariants of M.

properties of these functors

Lemma 16.
- *$M^{\mathfrak{g}}$ is the maixmal trivial $\mathfrak{g}$=submodule of M, and $(-)^{\mathfrak{g}}$ is right adjoint to the trivial $\mathfrak{g}$-module and hence left exact.*
- *$M_{\mathfrak{g}}$ is the largest quotient module of M that is trivial and $(-)_{\mathfrak{g}}$ is left adjoint to the trivial $\mathfrak{g}$-module and therefore right exact functor.*

Proof. If there is any $\mathfrak{g}$-submodule N of M that is trivial, then $\mathfrak{g}N = 0$ and hence $N \subseteq M^{\mathfrak{g}}$. Furthermore, we have the natural isomorphism

$$\begin{aligned} \mathrm{Hom}_k(V, M^{\mathfrak{g}}) &\to \mathrm{Hom}_{\mathfrak{g}}(V, M) \\ f &\to \iota \circ f \\ (g : V \to M^{\mathfrak{g}}) &\leftarrow (g : V \to M,) \end{aligned}$$

since $g(V) \subseteq M^{\mathfrak{g}}$.

For $M_{\mathfrak{g}}$, we have the natural isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}}(M, V) &\to \mathrm{Hom}_k(M_{\mathfrak{g}}, V) \\ f &\to (\overline{f} : M_{\mathfrak{g}} \to V) \\ f \circ \pi &\leftarrow f \end{aligned}$$

where $\pi : M \to M/\mathfrak{g}M$ is the quotient map and $\overline{f}$ is the induced map from f, since $\mathfrak{g}M \subseteq \ker f$. □

Using the Universal envoloping algebra, we can show that the category $\mathfrak{g} - \mathsf{Mod}$ has enough projectives and injectives.

Theorem 17. *Let $\mathfrak{g}$ be a Lie algebra. The category $\mathfrak{g} - \mathsf{Mod}$ is naturally isomorphic to the category $U\mathfrak{g} - \mathsf{Mod}$.*

Proof. The functors $\mathrm{Lie}(-) : k - alg \to Lie - algebras$ is right adjoint to the functor $U(-) : Lie - algebras \to k - alg$, so if M is a k-module and $E = \mathrm{End}_k(M)$ then we have

$$\mathrm{Hom}_{Lie}(\mathfrak{g}, \mathrm{Lie}(E)) \cong \mathrm{Hom}_{k-alg}(U\mathfrak{g}, \mathrm{End}_k(M)).$$

Using the fact that a $\mathfrak{g}$-module is the same as the lie algebra map $\mathfrak{g} \to \mathrm{Lie}(E)$ and a $U\mathfrak{g}$-module is the same as a k-algebra map $U\mathfrak{g} \to End_k(M)$, this proves the result. □

Therefore, we can define the left and right derived functors

Definition 18. Let M be a $\mathfrak{g}$-module. We write $H_*(\mathfrak{g}, M)$ for the left derived functors $L_*(-)_{\mathfrak{g}}(M)$ of $-_{\mathfrak{g}}$ and call them the homology groups of $\mathfrak{g}$ with coefficients in M.

We write $H^*(\mathfrak{g}, M)$ for the right derived functors $R^*(-^{\mathfrak{g}})(M)$ of $-^{\mathfrak{g}}$ and call them the cohomology groups of $\mathfrak{g}$ with coefficients in M.

Remark 19. If M is a $\mathfrak{g}$-module, then M becomes a $U\mathfrak{g}$-module by the action

$$(x_1 \otimes \cdots \otimes x_n) \cdot m = (x_1(x_2(\cdots(x_n m))\cdots)).$$

We can go the other way as well, by treating a $U\mathfrak{g}$-module M as a $\mathfrak{g}$-module just by restricting to the degree 1 terms, which is just $\mathfrak{g}$. So hence, if M and N are $\mathfrak{g}$-modules, then we have the natural isomorphism $F : \mathrm{Hom}_{\mathfrak{g}}(N, M) \to \mathrm{Hom}_{U\mathfrak{g}}(N, M)$ defined as taking $f \in \mathrm{Hom}_{\mathfrak{g}}(N, M)$ to f treated as a $U\mathfrak{g}$-module homomorphism. (this is well-defined from the action above). Furthermore, the inverse map F^{-1} takes $g \in \mathrm{Hom}_{U\mathfrak{g}}(N, M)$ to $\mathrm{Hom}_{\mathfrak{g}}(N, M)$ by just treating it as a $\mathfrak{g}$-module homomorphism, which is well-defined, since if

$$g(x_1 \otimes \cdots \otimes x_n) \cdot m) = x_1 \otimes \cdots \otimes x_n \cdot g(m),$$

for $x_i \in \mathfrak{g}$, then it works for when $n = 1$.

Connecting with Tor and Ext, we have

Theorem 20. *Let M be a $\mathfrak{g}$-module. Then*

$$\begin{aligned} H_*(\mathfrak{g}, M) &\cong \mathrm{Tor}_*^{U\mathfrak{g}}(k, M) \\ H^*(\mathfrak{g}, M) &\cong \mathrm{Ext}_{U\mathfrak{g}}^*(k, M). \end{aligned}$$

Proof. We have a augmentation map $\varepsilon : U\mathfrak{g} \to k$ by sending the image of $\mathfrak{g}$ to zero. So hence, $\ker \varepsilon$ is generated by elements in $\mathfrak{g}$. Hence we have

$$k \cong U\mathfrak{g}/\ker\varepsilon = U\mathfrak{g}/\mathfrak{g}(U\mathfrak{g}) \cong (U\mathfrak{g})_{\mathfrak{g}}.$$

Since all of these functors are universal delta functors, to show they are isomorphic, it suffices to show $(-)_{\mathfrak{g}} \cong k \otimes_{U\mathfrak{g}} -$ and $\mathrm{Hom}_{U\mathfrak{g}}(k, -) \cong (-)^{\mathfrak{g}}$. So for any $\mathfrak{g}$-module M, we have

$$k \otimes_{U\mathfrak{g}} M \cong (U\mathfrak{g}/\ker\varepsilon) \otimes_{U\mathfrak{g}} M \cong M/\ker\varepsilon M \cong M/\mathfrak{g}M = M_{\mathfrak{g}}.$$

and we have

$$\mathrm{Hom}_{U\mathfrak{g}}(k, M) \cong \mathrm{Hom}_{\mathfrak{g}}(k, M) \cong M^{\mathfrak{g}}.$$

(where the last isomorphism came from the natural map $T : \mathrm{Hom}_{\mathfrak{g}}(k, M) \to M^{\mathfrak{g}}$ defined as

$$T(g) = g(1),$$

□

2.2. **The Chevalley-Eilenberg Complex.** For this section, let $\mathfrak{g}$ be a lie algebra over a field k (or more generally, $\mathfrak{g}$ is a free k-module if k is just a commutative ring)

Definition 21. Let $\Lambda^p\mathfrak{g}$ denote the *pth*-exterior product of the k-module $\mathfrak{g}$. Denote $V_p(\mathfrak{g}) = U\mathfrak{g} \otimes_k \Lambda^p\mathfrak{g}$, which is a free left $U\mathfrak{g}$-module. By convention $\Lambda^0\mathfrak{g} = k$, and $\Lambda^1\mathfrak{g} = \mathfrak{g}$, so $V_0 = U\mathfrak{g}$ and $V_1 = U\mathfrak{g} \otimes_k \mathfrak{g}$. Define the augmentation map $\varepsilon : V_0 = U\mathfrak{g} \to k$ defined as sending $\mathfrak{g}$ to zero, i.e. $\ker\varepsilon = \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \cdots$. Also, define $d : V_1(\mathfrak{g}) \to V_0(\mathfrak{g})$ as the product map $d(u \otimes x) = ux$. Furthermore, for $p \geq 2$ let $d : V_p(\mathfrak{g}) \to V_{p-1}(\mathfrak{g})$ as

$$d(u \otimes x_1 \wedge \cdots \wedge x_n) = \theta_1 + \theta_2,$$

where θ_1, θ_2 are defined as

$$\theta_1 = \sum_{i=1}^{p} (-1)^{i+1} ux_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge x_p;$$
$$\theta_2 = \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_p.$$

The complex $V_\bullet(\mathfrak{g})$ is called the Chevalley-Eilenberg complex.

This complex gives us a projective resolution $V_\bullet(\mathfrak{g}) \to k$, where we treat k with the trivial action.

Theorem 22. *$V_\bullet(\mathfrak{g}) \to k$ is a projective resolution.*

Proof. To prove this theorem, we use the PBW theorem on $U\mathfrak{g}$, to construct a bounded below and exhaustive filtration on the complex $V_\bullet(\mathfrak{g})$. This uses Koszul complexes as well. □

A consequence of this is the following theorem

Corollary 23. *If M is a right $\mathfrak{g}$-module, then the homology modules $H_*(\mathfrak{g}, M)$ are the homology of the chain complex,*

$$M \otimes_{U\mathfrak{g}} V_*(\mathfrak{g}) = M \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes_k \Lambda^*\mathfrak{g} = M \otimes_k \Lambda^*\mathfrak{g}.$$

If M is a left $\mathfrak{g}$-module, then the cohomology modules $H^*(\mathfrak{g}, M)$ are the cohomogy of the cochain complex

$$\mathrm{Hom}_{\mathfrak{g}}(V(\mathfrak{g}), M) = \mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g} \otimes \Lambda^*\mathfrak{g}, M) \cong \mathrm{Hom}_k(\Lambda^*\mathfrak{g}, M).$$

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