Homework 2 for 509, Homological algebra, Spring 2019
due TBD
Preliminary version

Problem 1. Let R be a commutative ring, and N be an R-module. Show that the following are equivalent:

(1) N is flat
(2) $\mathrm{Tor}_i^R(M,N)=0$ for any $i>0$ and any R-module M
(3) $\mathrm{Tor}_1^R(M,N)=0$ for any R-module M
(4) $\mathrm{Tor}_1^R(R/I,N)=0$ for any ideal $I\subset R$.

Problem 2. Let $(R,\mathfrak{m},k)$ be a commutative Noetherian local ring and M be a finitely generated R-module. Prove that the following are equivalent:

(1) M is free
(2) M is flat
(3) The map $\mathfrak{m}\otimes_R M\to R\otimes_R M=M$ induced by the embedding $\mathfrak{m}\subset R$ is injective
(4) $\mathrm{Tor}_1(k,M)=0$.

Problem 3. Let R be a Frobenius algebra over a field k. Show that the global dimension of R is either zero or infinity.

Problem 4. Let k be a field of positive characteristic p.

(1) Show that the group algebra kG for a finite group G is Frobenius (in fact, symmetric).
(2) Show that the restricted enveloping algebra $\mathfrak{u}(\mathfrak{gl}_n)$ is Frobenius.

Problem 5. Let $\underline{x}=(x_1,x_2,\dots,x_n)$ be a sequence of elements in a commutative ring R, and let $K(\underline{x})=K(x_1)\otimes\ldots\otimes K(x_n)$ be the corresponding Koszul complex.

(1) Prove the explicit formula for the differential in $K(\underline{x})$.
(2) Show that $K(\underline{x})$ is a graded commutative DGA and, moreover, that $K(\underline{x})\simeq\Lambda^*(V)$ where $V=\bigoplus Rx_i$.